Formal Analysis of Unmanned Aerial Vehicles Using Higher-Order-Logic Theorem Proving

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The continuous dynamics of unmanned aerial vehicles (UAVs) are generally modeled as a set of differential equations. Traditionally, these continuous dynamics of UAVs are analyzed using paper-and-pencil proof and computer-based testing or simulations to study the performance, stability, and various other control characteristics of the aircraft flying in the air. However, these techniques suffer from their inherent limitations such as human error proneness, sampling-based analysis, approximations of the mathematical results, and the usage of unverified algorithms. Thus, these methods cannot be trusted when considering the utility of UAVs in many safety-critical applications. To overcome the limitations of the aforementioned techniques, it is proposed to use higher-order-logic theorem proving for formally analyzing the continuous dynamics of UAVs. In particular, a formalization of complex-valued matrices in higher-order logic is provided using the HOL Light theorem prover, which is in turn used for the formalization of the navigation's and aircraft's body-fixed frames, as well as their associated transformations. Formal reasoning support is also provided for analyzing the multiple-input/multiple-output systems, which are in turn used for formally analyzing the continuous dynamics of UAVs using HOL Light. For illustration, we use the current proposed framework for the formal stability analysis of the CropCam UAV using HOL Light.

I. Introduction

UNMANNED aerial vehicles (UAVs) [1] are widely being used in many domains, ranging from civilian [2] to military [3] applications, such as rescue missions [4], transportation [2], remote sensing [5] and surveillance [6], etc. They do not require any pilot aboard and are operated using a built-in computer system or a human operator on the ground to perform the flight control, navigation, and guidance operations. According to the economic report of year 2013 released by the Association for Unmanned Vehicle Systems International, the integration of unmanned aircraft systems (UASs) into the National Airspace System (NAS) will contribute an amount of more than 80 billion U.S. dollars to the U.S. economy from 2015 to 2025 [7]. This shows the essence of the unmanned aircraft industry in the NAS, which is significantly contributing to the economic development and playing a vital role in enhancing the safety aspects.

The dynamical analysis of an unmanned aerial vehicle incorporates the influence of different forces on the speed and attitude of the aircraft with respect to time and is widely used to study the performance, stability, and various other control characteristics of the aircraft flying in the air. This dynamical analysis requires modeling its continuous dynamics as general aircraft equations of motion that are based on the force, moment, kinematics, and navigation equations [8]. Moreover, this requires modeling a coordinate system that describes the aircraft motion in various frames and axes (i.e., navigation's and aircraft's body-fixed frames), and thus captures the position and movement of the aircraft [9]. Next, these aircraft equations, which are modeled as a system of differential equations, are solved in the frequency domain to either obtain the corresponding transfer function or the frequency response, or their solutions in the time domain using the Laplace transform [10]. These results can be further used for analyzing various characteristics, such as the stability and the control, of UAVs.

Traditionally, the dynamic analysis of these UAVs is conducted using the paper-and-pencil proof [11] and computer-based simulation methods [12]. However, the paper-and-pencil proof method is prone to error due to highly involved human manipulation: especially when dealing with larger systems exhibiting the complex dynamics. Similarly, the computer-based simulation provides approximate results due to the usage of finite precision of computer arithmetic, and thus compromises the accuracy of the analysis. Moreover, several unverified numerical algorithms are used in the associated tools. Thus, considering the safety-critical nature of UAVs, these conventional techniques should not be completely relied upon because this may lead to disastrous consequences. For example, according to the 2004 report by the U.S. Department of Transportation, 25% of the accidents happened to U.S. Army aircraft [Shadow 200 (RQ-7) unmanned] due to the failure of the tactical automated landing system [13]. A detailed account of accidents to UAVs and their reasons can be found in Ref. [13]. A rigorous analysis of these unmanned aircraft could have avoided such accidents.

Formal methods [14] have been used to overcome the aforementioned inaccuracy limitations for analyzing UAVs. Model checking [15] involves the development of a state-space-based model of the underlying system and the formal verification of its intended properties that are specified in temporal logic. It has been used (e.g., Refs. [16–18]) for analyzing UAVs. However, this kind of analysis involves the discretization of the continuous-time models, and thus compromises the accuracy of the corresponding analysis. Moreover, it also suffers from the infamous state-space explosion problem [19]. Higher-order-logic theorem proving [20] is a computer-based mathematical analysis method that requires developing a mathematical model of the given system in higher-order logic and the formal verification of its intended behavior as a mathematically specified property based on mathematical reasoning within the sound core of a theorem prover. The involvement of a formal model (specified in the expressive higher-order logic) and its associated formally specified properties, along with the sound nature of theorem proving, ensures the accuracy and completeness of the analysis. It has also been

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extensively used for analyzing UAVs (e.g., Refs. [21–23]). However, to the best of our knowledge, theorem proving has never been used for the analysis of the continuous dynamics of UAVs using the Laplace transform, which is the main scope of the current paper.

In this paper, we propose to use higher-order-logic theorem proving for formally analyzing UAVs. In particular, we provide a theorem-proving-based framework for formally analyzing the continuous dynamics of UAVs. We formalize various coordinate frames, such as navigation's and aircraft's body-fixed frames, which require the notion of the complex-valued matrices, which are formalized as a part of our proposed framework. We also formally verify the transformation between these frames. Moreover, we formalize the aircraft equations of motion capturing the continuous dynamics of UAVs, which are modeled as a set of linear differential equations. Finally, we extend the theory of the Laplace transform by providing a support for analyzing the multiple-input/multiple-output (MIMO) systems, which is further used for formally verifying the frequency-domain solutions and transfer function, as well as the time-domain solutions of the equations of motion of UAVs using the HOL Light theorem prover. Moreover, we use our proposed framework for the stability analysis of a CropCam UAV [8] using HOL Light. It is important to note that our proposed framework can be directly used for performing the Laplace transform-based analysis of the linear continuous dynamics of UAVs, i.e., for the case where the aircraft equations of motion are modeled using linear differential equations. However, the UAVs exhibiting nonlinear dynamical behavior cannot be directly analyzed, and thus the nonlinear differential equations modeling their continuous dynamics have to be first linearized and then our proposed framework can be used for their Laplace transform-based analysis.

The rest of the paper is organized as follows: We provide some related work in Sec. II. Section III presents an introduction about theorem proving and the HOL Light theorem prover. Section IV provides our proposed framework for the formal analysis of UAVs using HOL Light. Section V presents the formalization of the complex matrices that are required for formally analyzing UAVs. It also provides the formalization of various coordinate frames and the formal verification of their transformation, which ensures the correct orientation and position of UAVs. We present the formalization of the aircraft equations of motion (i.e., the longitudinal and the lateral–directional equations of motion), capturing the continuous dynamics of UAVs and the formal verification of their frequency and the time-domain solutions in Sec. VI. Section VII provides the formal stability analysis of CropCam UAV using HOL Light. Finally, Sec. VIII concludes the paper.

II. Related Work

Formal methods (in particular, model checking) have been widely used for formally analyzing UAVs. Groza et al. [17] used the hybrid logic model checker (HLMC) to formally analyze UAVs. The authors developed a formal goal structuring notation (GSN) model of the vehicle and used description logic to identify the assurance deficits in the corresponding GSN model. Finally, the identified flaws are verified using properties specification in the HLMC. Similarly, Seibel et al. [24] proposed a hybrid automata-based approach for mission planning of the rotary-wing UAVs, which is based on the reachability analysis techniques, and thus enables the verification of safety and timeliness requirements by avoiding the undesirable behaviors, such as violation of trajectory margins. Guzey [18] also used hybrid automata to model the controllers for the fixed-wing UAVs, which are used to maintain their predefined formation and keep them on track to their goal location.

Karimoddini et al. [16] proposed a hierarchical approach for modeling and control design of an unmanned helicopter. The authors modeled its control structure as a set of three layers in hierarchical form: namely, motion planning, regulation, and supervision layers, where each layer is modeled as an individual hybrid automaton. Finally, a compositional operator is developed for the synchronization of all these layers. Schumann et al. [25] developed a framework, titled R2U2, for the runtime verification of the onboard UASs, which provides the runtime monitoring of the properties regarding security threats and their diagnoses. R2U2 uses the linear temporal logic and Bayesian networks for property monitoring and security threat diagnostics, respectively.

Webster et al. [26] used the Simple Promela Interpreter (SPIN) model checker for the formal certification of an autonomous unmanned aircraft system. The authors developed a fundamental UAS control system model in Process or Protocol Meta Language (PROMELA) (which is language for the SPIN model checker) and verified it against a selected subset of rules defined by the Civil Aviation Authority using the SPIN model checker. The authors also developed a probabilistic model, incorporating the probabilistic aspects, and verified the same set of rules using the PRISM model checker. Similarly, the authors modeled the UAS control system using the autonomous agent language Gwendolen and performed its formal analysis using the agent model checker Agent Java Pathfinder (AJPF) [27]. Finally, a comparison of the results obtained by the aforementioned three approaches is presented, which leads to a full certification of the UAS.

All the model-checking-based analyses presented earlier in this paper consider the discrete time models of the aircraft in the form of automata and are unable to capture their continuous dynamics in true form. Moreover, model checking suffers from its inherent state-space explosion problem, and thus is not well suited for analyzing systems exhibiting the continuous dynamics, which generally leads to large models.

Theorem proving has also been used for the formal analysis of UAVs. Munoz et al. [21] used the Prototype Verification System (PVS) theorem prover for formally analyzing the extended well-clear boundaries of unmanned aircraft that express the capability of an aircraft to avoid collisions with other airborne traffic by keeping itself away from the other aircraft. Similarly, Munoz and Narkawicz [22] proposed a detect and avoid alerting logic for unmanned systems (known as DAIDALUS), which consists of the self-separation and alerting algorithms that are implemented on a detect and avoid concept, and thus provide situation awareness to UAS remote pilots. The authors also formally verified these algorithms using PVS. Narkawicz and Munoz [23] proposed a framework for the formal verification of the conflict detection algorithms for aircraft flying on arbitrary nonlinear trajectories using PVS. Similarly, Ghorbal et al. [28] proposed an approach based on hybrid theorem proving for formally analyzing the aerospace systems. The authors used their proposed approach for formally verifying the property of separation between two or more aircraft. Jasim and Veres [29] presented a model-based approach for formally verifying the stability of the robust attitude controller of a quadcopter using the MetiTarski theorem prover. Similarly, Denman et al. [30] used MetiTarski for formally verifying the properties of Nichols plots and used their proposed framework for formally analyzing the lateral autopilot of a model 24 Learjet subsonic business jet. Chen and Chen [31] formally verified a control algorithm for automatic landing of a helicopter using the Coq theorem prover. Similarly, Ma and Chen [32] used Coq for formally verifying the coordinate transformation matrices of the aircraft control system. The authors have used a real data type R for modeling various aerodynamics parameters and their associated transformation matrices. Carreno and Munoz [33] formally verified the correctness of an alerting algorithm for aircraft using the PVS theorem prover.

Ricketts et al. [34] presented the proof rules, capturing common reasoning patterns for modular construction and verification of the periodic sampled-data cyberphysical systems (CPSs). These systems are based on digital controllers that are running periodically. Moreover, the system exhibits continuous dynamical behavior in between executions of the controller. Finally, the authors used their proposed rules for formally verifying the quadcopter controllers, enforcing their safety properties, such as geofences. Malecha el al. [35] proposed a foundational framework, VeriDrone, to formally reason about CPSs. VeriDrone is a library developed in Coq containing theories ranging from real numbers: floating point numbers to differential equations. It expresses the properties of CPSs in linear temporal logic that is deeply embedded within Coq. Similarly, Chan et al. [36]

used VeriDrone for formally verifying the stability properties (i.e., Lyapunov and exponential stabilities) of CPSs. Loos et al. [37] formally verified the control policies for planar aircraft avoidance maneuvers and provided a proof of their safety using the KeYmaeraD theorem prover. Arechiga et al. [38] used KeYmaera for formally verifying the closed-loop properties of the control system. The authors also formally verified the safety of an intelligent cruise controller and a cooperative intersection collision avoidance system. However, none of the analyses based on theorem proving (presented earlier) provides the analysis of the continuous dynamics of UAVs using the Laplace transform, which is the main scope of this paper.

III. Preliminaries

This section provides a brief introduction about theorem proving and the HOL Light theorem prover.

A. Theorem Proving

Theorem proving [20] is concerned with the development of mathematical proofs using a computer program, commonly known as a theorem prover/proof assistant, which uses a small set of axioms, inference rules and hypothesis, and the already verified theorems to verify new theorems. Theorem provers have been extensively employed for the formalization (mathematical modeling and the development of the formal proofs) of the classical mathematics, such as the formalization of Euclidean space in the HOL Light theorem prover [39]; the formal proof of the Kepler conjecture [40], etc.; and for the formal verification of many software and hardware systems [41,42]. For example, we can certify a digital circuit by formally verifying the mathematical theorems capturing its various properties that are expressed in some appropriate logic, which can be propositional, first-order, or higher-order logic. Based on the decidability or undecidability of the underlying logic (i.e., propositional or higher-order logic), theorem proving can be automatic or interactive, respectively. For example, a computer program can automatically verify the theorems about sentences expressed in the propositional logic due to the decidability of this logic, whereas the higher-order logic is undecidable; thus, verifying sentences expressed in this logic requires explicit user guidance in an interactive manner.

B. HOL Light Theorem Prover

HOL Light [43] is a higher-order-logic proof assistant that ensures secure theorem proving using the Objective Categorical Abstract Machine Language (CAML) (known as OCaml) language, which is a variant of the strongly typed functional programming language Meta Language (ML) [44]. HOL Light users can interactively verify theorems by applying the available proof tactics and proof procedures. A HOL Light theory consists of types, constants, definitions, and theorems. HOL Light theories are built in a hierarchical fashion, and new theories can inherit the definitions and theorems of their parent theories. HOL Light consists of a rich set of formalized theories including sets, Boolean algebra, arithmetic, real numbers, multivariate calculus, and the Laplace transform, which are extensively used in our formalization. In fact, the availability of the multivariable calculus and the Laplace transform theories was the main motivation for the selection of the HOL Light theorem prover for the proposed framework. The Laplace transform has also been formalized in Coq [45] and Isabelle [46] theorem provers. However, its formalization in Coq does not contain the uniqueness of the Laplace transform that provides the solution of the linear differential equations in the time domain and is used, in our proposed framework, for analyzing the continuous dynamics of UAVs. Similarly, the formal library of the Laplace transform in Isabelle does not contain the formalization providing the analysis of MIMO systems.

IV. Proposed Framework

The proposed framework for formally analyzing the continuous dynamics of UAVs using the HOL Light theorem prover is depicted in Fig. 1. In the first step of the analysis, our framework accepts the coordinate frames and the set of linear differential equations modeling the continuous dynamics of UAVs from the user. The given coordinates are transformed to the corresponding model, using the navigation's and the aircraft's body-fixed frames, in higher-order logic. Similarly, the given set of the differential equations is transformed to the corresponding model in higher-order logic, i.e., the longitudinal and lateral–directional equations of motion. Next, we have to verify the transformation of the coordinate frames, which requires the notion of the complex matrices. Therefore, we have formalized complex matrices in HOL Light as a part of our proposed framework. Similarly, we have to verify various other properties based on the continuous dynamics of UAVs, which are usually expressed in terms of a transfer function, frequency response, and the frequency- and time-domain solutions of the set of differential equations. To carry out the verification process of these properties, we extend our formalization of the Laplace transform in HOL Light by providing reasoning support for formally analyzing the MIMO

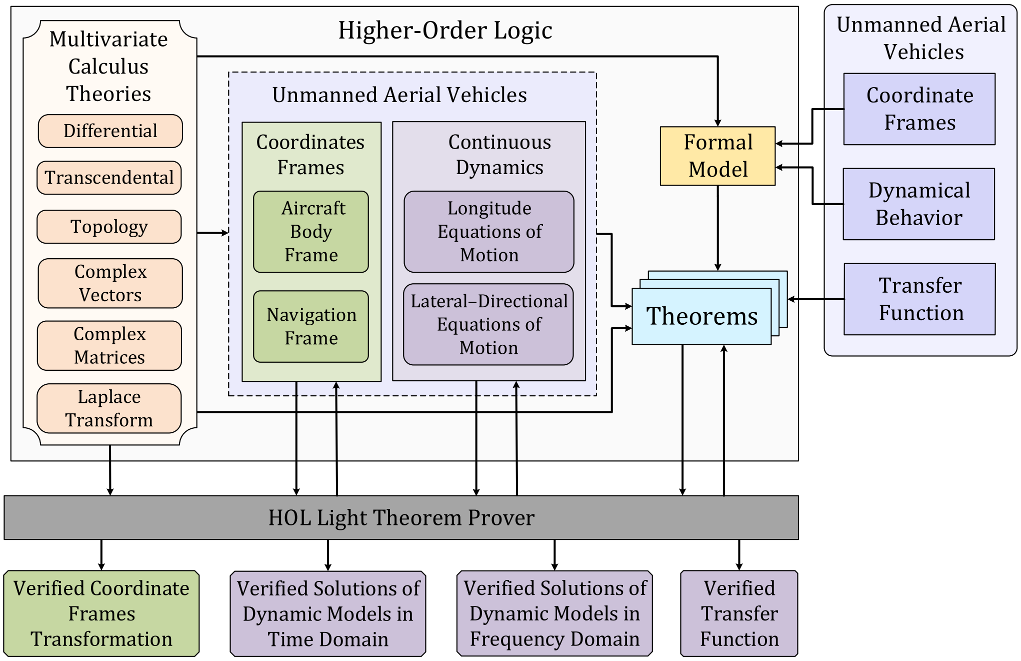

Fig. 1 Proposed framework.

systems, which is further used for formally analyzing the continuous dynamics of UAVs and the stability analysis of the CropCam UAV using HOL Light.

V. Formalization of the Coordinate Frames

Coordinate systems are used to capture the orientation and position of UAVs at any time instant. Two commonly used coordinate frames in the context of aircraft are navigation's and aircraft's body-fixed frames. The aircraft motion is usually described with respect to the navigation frame. It is oriented as north, east, and down (x_n, y_n, and z_n, respectively) and is attached to Earth's local tangent plane as shown in Fig. 2. Similarly, a right-handed orthogonal body coordinate frame (x_b, y_b, and z_b) is attached to the aircraft, as depicted in Fig. 2. The origin O_b of the aircraft body frame is located at the aircraft's center of mass. The positive x axis of the frame is directed forward along the aircraft's longitudinal axis. Similarly, the positive y axis is oriented along the right wing, and the positive z axis is perpendicular to the x and y axes and is pointing downward from the aircraft.

To facilitate the understanding of the paper for a non-HOL user, we present the higher-order-logic formalization of the coordinate frames of UAVs using standard mathematical notations rather than the HOL Light notations. The proof script for our formalization can be obtained from Ref. [47] for the readers who are interested in viewing the exact HOL Light formalization presented in this paper.

To model the respective coordinate frames, such as navigation's and aircraft's body-fixed frames, we require modeling a point, which captures the position and orientation of a UAV in a coordinate system. In HOL Light, we can use the available types (e.g., real R, complex C, and one-dimensional real-valued vector R^1) to abbreviate new types. Therefore, we use the feature of type abbreviation in HOL Light to define new types for various points as follows:

Definition V.1: points of a coordinate system:

new_type_abbrev ("one_dim_point",':($R^1 \rightarrow C$)')

new_type_abbrev ("timed_one_dim_point",':(one_dim_point $\times R^1$)')

new_type_abbrev ("two_dim_point",':(one_dim_point $\times$ one_dim_point)')

new_type_abbrev ("timed_two_dim_point",':(two_dim_point $\times R^1$)')

new_type_abbrev ("three_dim_point",':(one_dim_point $\times$ two_dim_point)')

new_type_abbrev ("timed_three_dim_point",':(three_dim_point $\times R^1$)')

The type timed_one_dim_point is a pair, capturing the orientation and position of a UAV in one-dimensional coordinate system that changes with time, where its second element models the time. To formalize the coordinate frames, we require the notion of the complex-valued matrices, which are formalized as a part of our proposed framework; their details can be found in Ref. [47].

Now, the navigation's and the aircraft's body-fixed coordinate frames are three-dimensional coordinates, which are modeled in HOL Light as follows:

Definition V.2: three-dimensional coordinates:

$$\vdash_{\text{def}} \forall\ \mathtt{x}\ \ \mathtt{y}\ \ \mathtt{z}\ \ \mathtt{t}.\ \ \textbf{three\_dim\_coord\_sys}$$

$$(((\mathtt{x},\mathtt{y},\mathtt{z}),\mathtt{t}):\mathtt{timed\_three\_dim\_point}) = \begin{bmatrix} \mathtt{x(t)} \\ \mathtt{y(t)} \\ \mathtt{z(t)} \end{bmatrix}$$

The function three_dim_coord_sys accepts a variable of data-type timed_three_dim_point and returns a three-dimensional vector describing the corresponding coordinate frame.

A continuous rotation of the orientation of the navigation frame by Euler angles (angles defined by performing the rotation about the axes of three-dimensional right-handed coordinate system) transforms it to the aircraft's body-fixed frame. To formally verify this transformation, we first model the Euler angles in HOL Light using the feature of type abbreviation as follows:

Definition V.3: Euler angles:

new_type_abbrev ("theta", ':C')

new_type_abbrev ("phi", ':C')

new_type_abbrev ("psi", ':C')

new_type_abbrev ("euler_angles", ':(theta $\times$ phi $\times$ psi)')

We use the complex data type C for capturing the Euler angles because this choice allows us to formally model the stability of UAV, which is based on the placement of the poles (roots of characteristic equation) in the left half of the complex plane, as will be depicted in Sec. VII of the paper. The Euler angles and the corresponding frame transformations are depicted in Fig. 3. A rotation of yaw angle ψ about the z_0 axes transforms the navigation frame to intermediate frame 1, which defines the aircraft's heading. A further rotation of pitch angle θ about the new y_1 axis results into the transformation of intermediate frame 1 to intermediate frame 2. Finally, intermediate frame 2 is transformed to the aircraft's body-fixed frame by a rotation of roll angle ϕ about the new x_2 axis.

To formally verify these transformations, we require the rotation matrices for rolling, pitching, and yawing, capturing the relative orientations among various coordinate frames. They are formalized in HOL Light as follows:

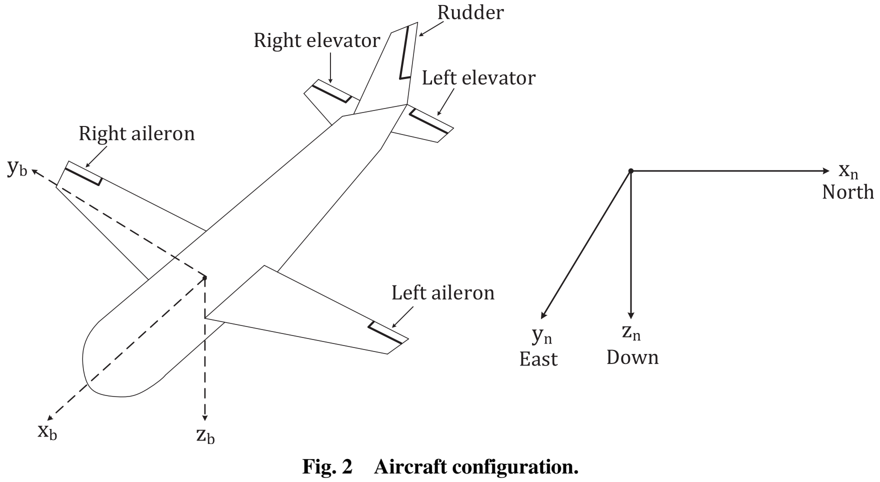

Fig. 2 Aircraft configuration.

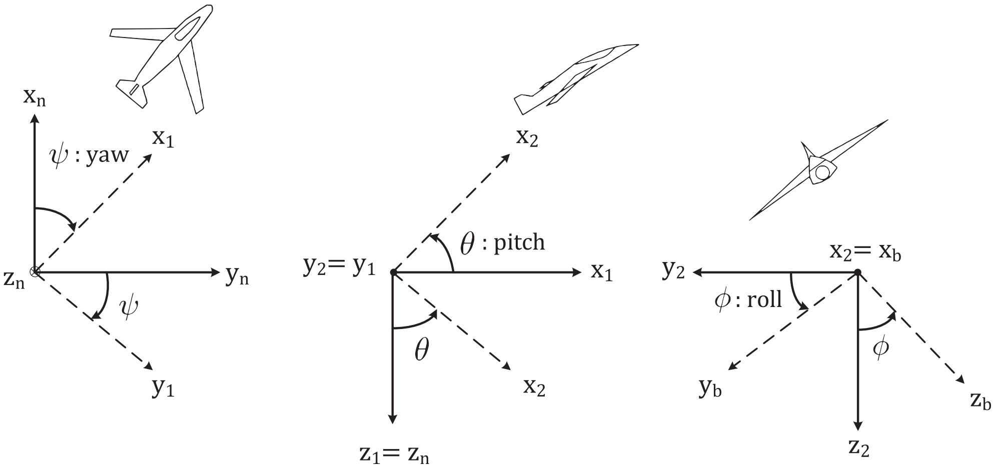

Fig. 3 Euler angles and frame transformation.

Definition V.4: rotation matrices:

$$\vdash_{\text{def}} \forall \texttt{theta psi phi}.\ \textbf{rot\_mat\_roll}\ (\texttt{theta},\texttt{psi},\texttt{phi}) = \begin{bmatrix} \texttt{Cx(\&1)} & \texttt{Cx(\&0)} & \texttt{Cx(\&0)} \\ \texttt{Cx(\&0)} & \texttt{ccos(phi)} & \texttt{csin(phi)} \\ \texttt{Cx(\&0)} & -\texttt{csin(phi)} & \texttt{ccos(phi)} \end{bmatrix}$$

$$\vdash_{\text{def}} \forall \texttt{phi psi theta}.\ \textbf{rot\_mat\_pitch}\ (\texttt{theta},\texttt{psi},\texttt{phi}) = \begin{bmatrix} \texttt{ccos(theta)} & \texttt{Cx(\&0)} & -\texttt{csin(theta)} \\ \texttt{Cx(\&0)} & \texttt{Cx(\&1)} & \texttt{Cx(\&0)} \\ \texttt{csin(theta)} & \texttt{Cx(\&0)} & \texttt{ccos(theta)} \end{bmatrix}$$

$$\vdash_{\text{def}} \forall \texttt{theta phi psi}.\ \textbf{rot\_mat\_yaw}\ (\texttt{theta},\texttt{psi},\texttt{phi}) = \begin{bmatrix} \texttt{ccos(psi)} & \texttt{csin(psi)} & \texttt{Cx(\&0)} \\ -\texttt{csin(psi)} & \texttt{ccos(psi)} & \texttt{Cx(\&0)} \\ \texttt{Cx(\&0)} & \texttt{Cx(\&0)} & \texttt{Cx(\&1)} \end{bmatrix}$$

where the HOL Light functions ccos and csin represent the complex-valued cosine and sine functions, respectively. Similarly, and and Cx typecast a natural number to real number and a real number to a complex number, respectively. Now, we formally verify the transformation from the navigation frame to intermediate frame 1 as the following HOL Light theorem:

Theorem V.1: navigation frame to intermediate frame 1:

$$\vdash_{\text{thm}} \forall \texttt{xn yn zn x1 y1 z1 theta phi psi t. trans\_navig\_frame\_inter\_frame\_one (x1,y1,z1)}$$

$$\texttt{(xn,yn,zn)(theta,phi,psi) t} \Leftrightarrow \left(\begin{bmatrix} \texttt{x1(t)} \\ \texttt{y1(t)} \\ \texttt{z1(t)} \end{bmatrix} = \begin{bmatrix} \texttt{xn(t)}*\texttt{ccos(psi)}+\texttt{yn(t)}*\texttt{csin(psi)} \\ -\texttt{xn(t)}*\texttt{csin(psi)}+\texttt{csin(psi)}+\texttt{yn(t)}*\texttt{ccos(psi)} \\ \texttt{zn(t)} \end{bmatrix} \right)$$

The function trans_navig_frame_inter_frame_one models the corresponding transformation. The verification of the preceding theorem is based on the properties of the vectors, complex matrices, and transcendental functions along with some complex arithmetic reasoning. We also verified the transformations from intermediate frame 1 to intermediate frame 2 and intermediate frame 2 to the aircraft's body-fixed frame as the following two theorems:

Theorem V.2: intermediate frame 1 to intermediate frame 2:

$$\vdash_{\text{thm}} \forall \texttt{x2 y2 z2 x1 y1 z1 theta phi psi t. trans\_inter\_frame\_one\_inter\_frame\_two (x2,y2,z2)}$$

$$\texttt{(x1,y1,z1)(theta,phi,psi) t} \Leftrightarrow \left(\begin{bmatrix} \texttt{x2(t)} \\ \texttt{y2(t)} \\ \texttt{z2(t)} \end{bmatrix} = \begin{bmatrix} \texttt{x1(t)} * \texttt{ccos(theta)} - \texttt{z1(t)} * \texttt{csin(theta)} \\ \texttt{y1(t)} \\ \texttt{x1(t)} * \texttt{csin(theta)} + \texttt{z1(t)} * \texttt{ccos(theta)} \end{bmatrix} \right)$$

Theorem V.3: intermediate frame 2 to aircraft's body-fixed frame:

$$\vdash_{\text{thm}} \forall \texttt{xb yb zb x2 y2 z2 theta phi psi t. trans\_inter\_frame\_two\_aircraft\_fixed (xb,yb,zb)}$$

$$\texttt{(x2,y2,z2)(theta,phi,psi) t} \Leftrightarrow \left(\begin{bmatrix} \texttt{xb(t)} \\ \texttt{yb(t)} \\ \texttt{zb(t)} \end{bmatrix} = \begin{bmatrix} \texttt{x2(t)} \\ \texttt{y2(t)} * \texttt{ccos(phi)} + \texttt{z2(t)} * \texttt{csin(phi)} \\ -\texttt{y2(t)} * \texttt{csin(phi)} + \texttt{z2(t)} * \texttt{ccos(phi)} \end{bmatrix} \right)$$

The functions trans_inter_frame_one_inter_frame_two and trans_inter_frame_two_aircraft_fixed in Theorems V.2 and V.3 represent the corresponding transformations. The proof process of these theorems is very similar to that of Theorem V.1.

Next, in order to verify the transformation of the navigation frame to the aircraft's body-fixed frame, we first model the direction cosine matrix in HOL Light as follows:

Definition V.5: direction cosine matrix:

$$\vdash_{\text{def}} \forall \texttt{psi phi theta}.\ \textbf{direc\_cos\_mat}(\texttt{theta},\texttt{phi},\texttt{psi}) = \begin{bmatrix} \texttt{A} & \texttt{B} & \texttt{C} \\ \texttt{D} & \texttt{E} & \texttt{F} \\ \texttt{G} & \texttt{H} & \texttt{I} \end{bmatrix}$$

where

```
A = ccos (theta) *ccos (psi)
B = ccos (theta) * ccos (psi)
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  C = - csin (theta)
  D = csin (phi) * csin (theta) * ccos (psi) - ccos (phi) * csin (psi)
  E = csin (phi) * csin (theta) * csin (psi) + ccos (phi) * ccos (psi)
  F = csin (phi) * ccos (theta)
  G = ccos (phi) * csin (theta) * ccos (psi) + csin (phi) * csin (psi)
  H = ccos (phi) * csin (theta) * csin (psi) - csin (phi) * ccos (psi)
  I = ccos (phi) * ccos (theta)
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Now, we use the direction cosine matrix (Definition V.5) to formalize the transformation from the navigation frame to the aircraft's body-fixed frame in HOL Light as follows:

Definition V.6: navigation frame to aircraft's body-fixed frame:

```
⊢ ∀ xb yb zb xn yn zn theta phi psi t.
     trans_aircraft_fixed_navig_frame (xb,yb,zb) (xn,yn,zn) (theta,phi,psi) t ⇔
        (three_dim_coord_sys ((xb,yb,zb), t) = direc_cos_mat (theta,phi,psi) ** three_dim_coord_sys ((xn,yn,zn), t))
```

Next, we verify the transformation from the navigation frame to aircraft's body-fixed frame (Definition V.6) as the following HOL Light theorem:

Theorem V.4: navigation frame to aircraft's body-fixed frame:

```
⊢ ∀ xb yb zb x2 y2 z2 x1 y1 z1 xn yn zn theta phi psi t.
  Assumption A1: trans_navig_frame_inter_frame_one (x1,y1,z1) (xn,yn,zn) (theta,phi,psi) t ∧
  Assumption A2: trans_inter_frame_one_inter_frame_two (x2,y2,z2) (x1,y1,z1) (theta,phi,psi) t ∧
  Assumption A3: trans_aircraft_fixed_inter_frame_two (xb,yb,zb) (x2,y2,z2) (theta,phi,psi) t
     ⇒ trans_aircraft_fixed_navig_frame (xb,yb,zb) (xn,yn,zn) (theta,phi,psi) t
```

Assumption A1 presents the transformation from the navigation frame to intermediate frame 1. Similarly, Assumptions A2 and A3 provide the transformations from intermediate frame 1 to intermediate frame 2 and intermediate frame 2 to the aircraft's body-fixed frame, respectively.[§] The conclusion models the transformation of the navigation frame to the aircraft's body-fixed frame. The verification of Theorem V.4 is mainly based on Theorems V.1, V.2, and V.3, as well as the properties of vector and complex matrices. This concludes our formal verification of the transformation of various coordinate frames. The verified results presented in this section can be used to reason about the correct orientation and position of UAVs at any time instant. The details about the formalization can be found in Ref. [47].

VI. Formal Analysis of the Continuous Dynamics of UAVs

In this section, we present the formal analysis of the continuous dynamics of UAVs, which includes the formalization of the linear differential equations-based longitudinal and lateral–directional equations of motion, which capture the aircraft's dynamics, and the verification of their frequency- and time-domain solutions. These equations are generally obtained by applying the mathematical and physical laws regarding force and inertia, and thus provide a relationship between the input and output variables of the aircraft as shown in Fig. 4. Because the number of input and output variables of the underlying system is greater than one, it is considered a MIMO system. To formally analyze these equations in HOL Light, first, we need to incorporate various parameters involved in these equations, such as force, velocities, moment of inertia, etc.

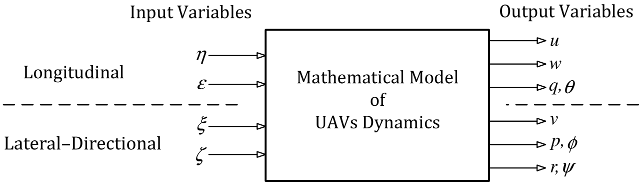

Fig. 4 UAV input–output relationship.

We first model the angular and linear disturbance velocities in HOL Light using the feature of type abbreviation as follows:

Definition VI.1: angular disturbance velocities:

```
  new_type_abbrev ("p", `:R^1 → C`)
  new_type_abbrev ("q", `:R^1 → C`)
  new_type_abbrev ("r", `:R^1 → C`)
  new_type_abbrev ("angular_distur_vel", `:(theta × phi × psi)')
```

Definition VI.2: linear disturbance velocities:

```
  new_type_abbrev ("u", `:R^1 → C`)
  new_type_abbrev ("v", `:R^1 → C`)
  new_type_abbrev ("w", `:R^1 → C`)
  new_type_abbrev ("linear_distur_vel",`:(theta × phi × psi)')
```

where each component of the angular and linear disturbance velocities is a function of type $R^1 \to C$. Similarly, the aerodynamics stability derivatives represent constant values for the linear longitudinal equations of motion of UAVs and are of type C, as given in Table 1. These derivatives for the axial (drag) force, normal (lift) force, and pitching moment are modeled using type abbreviations as follows:

Definition VI.3: Aerodynamics Stability Derivatives:

```
  new_type_abbrev ("aero_sta_deriv_axial_force",`:(Xu × Xv × Xw × Xwd × Xq × Xeta × Xtau)')
  new_type_abbrev ("aero_sta_deriv_normal_force", `:(Zu × Zw × Zwd × Zq × Zeta × Ztau)')
  new_type_abbrev ("aero_sta_deriv_pitch_moment", `:(Mu × Mw × Mwd × Mq × Meta × Mtau)')
```

Similarly, we model the linear velocity and moment of inertia of UAVs using HOL Light's type abbreviation feature as follows:

Definition VI.4: linear velocity:

```
  new_type_abbrev ("Ue", `:C`)
  new_type_abbrev ("Ve", `:C`)
  new_type_abbrev ("We", `:C`)
  new_type_abbrev ("linear_vel", `:(Ue × Ve × We)')
```

Definition VI.5: moment of inertia:

```
  new_type_abbrev ("Ix", `:C`)
  new_type_abbrev ("Iy", `:C`)
  new_type_abbrev ("Iz", `:C`)
  new_type_abbrev ("mom_of_inert", `:(Ix × Iy × Iz)')
```

where `Ue`, `Ve`, and `We` model the axial, lateral, and normal velocities, respectively. Similarly, `Ix`, `Iy`, and `Iz` present the x, y, and z components of the moment of inertia, respectively.

[§] The formal definitions of transformations from the navigation frame to intermediate frame 1, intermediate frame 1 to intermediate frame 2, and intermediate frame 2 to the aircraft's body-fixed frame can be found in the proof script in Ref. [47].

Table 1 Aerodynamics stability derivatives and their data types

Symbol	Type
Xu	ℂ
Xw	ℂ
Xwd	ℂ
Xq	ℂ
Xv	ℂ
Xtau	ℂ
Xeta	ℂ
Zw	ℂ
Zwd	ℂ
Zq	ℂ
Zu	ℂ
Ztau	ℂ
Zeta	ℂ
Mu	ℂ
Mw	ℂ
Mwd	ℂ
Mq	ℂ
Meta	ℂ
Mtau	ℂ

Generally, the motion of a UAV is described by decoupled equations of motion [10] such as longitudinal and lateral–directional equations of motion. The longitudinal equations of motion for the aircraft are described by the axial force X, the normal force Z, and the pitching moment M; and they are mathematically expressed as follows [8,10]:

$$\begin{aligned} m\dot{u} - X_u u - X_{\dot{w}}\dot{w} - X_w w - (X_q - mW_e)q \\ + mg\theta\cos\theta_e = X_\eta \eta + X_\tau \tau \\ - Z_u u + (m - Z_{\dot{w}})\dot{w} - Z_w w - (Z_q + mU_e)q \\ + mg\theta\sin\theta_e = Z_\eta \eta + Z_\tau \tau \\ M_u u - M_{\dot{w}}\dot{w} - M_w w + I_y \dot{q} - M_q q = M_\eta \eta + M_\tau \tau \end{aligned}$$

where $\dot{u}$, $\dot{w}$, and $\dot{q}$ represent the first-order derivatives of the linear and angular disturbance velocities, respectively. Similarly, m, g, and θ_e are the total mass of the UAV, acceleration due to gravity, and steady pitch attitude of the UAV, respectively. The variables η and τ model the elevator angle and thrust, respectively. Under the conditions $q(t) = \dot{\theta}(t)$ and $\tau(t) = 0$, the preceding longitudinal equations of motion of the UAV become [10]

$$\begin{aligned} m\dot{u} - X_u u - X_{\dot{w}}\dot{w} - X_w w - (X_q - mW_e)\dot{\theta} \\ + mg\theta\cos\theta_e = X_\eta \eta \\ -Z_u u + (m - Z_{\dot{w}})\dot{w} - Z_w w - (Z_q + mU_e)\dot{\theta} \\ + mg\theta\sin\theta_e = Z_\eta \eta \\ M_u u - M_{\dot{w}}\dot{w} - M_w w + I_y \ddot{\theta} - M_q \dot{\theta} = M_\eta \eta \end{aligned} \tag{2}$$

To model the preceding set of linear differential equations, we first formalize a linear differential equation of order n in HOL Light as follows:

Definition VI.6: differential equation of order n

```
⊢def ∀ n lst f t. diff_eq_n_order n lst f t = vsum
(0..n) (λk. EL k lst * higher_vector_derivative
k f t)
```

The function `diff_eq_n_order` accepts the order of the differential equation `n`, a list of constant coefficients `lst`, a differentiable function `f`, and the differentiation variable `t`. It uses the functions `vsum n f` and `EL k lst`, which return the vector summation

$$\sum_{i=0}^{n} f_i$$

and the kth element of a list `lst`, respectively, to generate the differential equation corresponding to the given parameters.

Now, the first differential equation of the longitudinal equations of motion [Eq. (2)] is formalized as follows:

Definition VI.7: first longitudinal equation of motion:

```
⊢def ∀ Xv Xw Xwd Xq Xeta Xtau Xu m. lst_u_lon_eq_
mot_fst m (Xu,Xv,Xw,Xwd,Xq,Xeta,Xtau) = [-Xu;
Cx m]
⊢def ∀ Xu Xv Xq Xeta Xtau Xw Xwd. lst_w_lon_eq_
mot_fst (Xu,Xv,Xw,Xwd,Xq,Xeta,Xtau) = [Xw; Xwd]
⊢def ∀ Xu Xv Xw Xwd Xeta Xtau Ue Ve g thetae Xq m We.
lst_theta_lon_eq_mot_fst m g thetae
        (Xu,Xv,Xw,Xwd,Xq,Xeta,Xtau) (Ue,Ve,We)
= [-(Cx m * Cx g * ccos thetae); Xq - Cx m * We]
⊢def ∀ Xu Xv Xw Xwd Xq Xtau Xeta. lst_eta_lon_eq_
mot_fst (Xu,Xv,Xw,Xwd,Xq,Xeta,Xtau) = [Xeta]
⊢def ∀p q r v u w m g thetae Ue Ve We theta Xu Xv Xw Xwd
Xq Xeta Xtau eta t. lon_eq_mot_fst
        (Xu,Xv,Xw,Xwd,Xq,Xeta,Xtau) (Ue,Ve,We)
m g thetae (p,q,r) (u,v,w) theta eta t ⇔
      diff_eq_n_order 1 (lst_u_lon_eq_mot_fst m
(Xu,Xv,Xw,Xwd,Xq,Xeta,Xtau)) u t -
      diff_eq_n_order 1 (lst_w_lon_eq_mot_fst
(Xu,Xv,Xw,Xwd,Xq,Xeta,Xtau)) w t -
      diff_eq_n_order 1 (lst_theta_lon_eq_mot_
fst m g thetae (Xu,Xv,Xw,Xwd,Xq,Xeta,Xtau) (Ue,
Ve,We)) theta t =
      diff_eq_n_order 0 (lst_eta_lon_eq_mot_fst
(Xu,Xv,Xw,Xwd,Xq,Xeta,Xtau)) eta t
```

where the function `lon_eq_mot_fst` accepts the function variables `u`, `w`, `theta`, and `eta`; and the lists of coefficients `lst_u_lon_eq_mot_fst`, `lst_w_lon_eq_mot_fst`, `lst_theta_lon_eq_mot_fst`, and `lst_eta_lon_eq_mot_fst`; and returns the corresponding differential equation. Similarly, we model the other two differential equations of the longitudinal equations of motion [Eq. (2)] as follows:

Definition VI.8: second longitudinal equation of motion:

```
⊢def ∀ Zw Zwd Zq Zeta Ztau Zu. lst_u_lon_eq_
mot_snd (Zu,Zw,Zwd,Zq,Zeta,Ztau) = [-Zu]
⊢def ∀ Zu Zq Zeta Ztau Zw m Zwd. lst_w_lon_eq_
mot_snd m (Zu,Zw,Zwd,Zq,Zeta,Ztau) = [Zw; -(Cx m
- Zwd)]
⊢def ∀ Zu Zw Zwd Zeta Ztau Ue We g thetae Zq m Ve.
lst_theta_lon_eq_mot_snd m g thetae
      (Zu,Zw,Zwd,Zq,Zeta,Ztau) (Ue,Ve,We) = [- (Cx
m * Cx g * csin thetae); Zq + Cx m * Ve]
⊢def ∀ Zu Zw Zwd Zq Ztau Zeta. lst_eta_lon_eq_
mot_snd (Zu,Zw,Zwd,Zq,Zeta,Ztau) = [Zeta]
⊢def ∀p q r v u w m g thetae Ue Ve We theta Zu Zw Zwd Zq
Zeta Ztau eta t. lon_eq_mot_snd
      (Zu,Zw,Zwd,Zq,Zeta,Ztau) (Ue,Ve,We) m g
thetae (p,q,r) (u,v,w) theta eta t ⇔
      diff_eq_n_order 0 (lst_u_lon_eq_mot_snd
(Zu,Zw,Zwd,Zq,Zeta,Ztau)) u t -
      diff_eq_n_order 1 (lst_w_lon_eq_mot_snd
(Zu,Zw,Zwd,Zq,Zeta,Ztau)) w t -
```

diff_eq_n_order 1 (lst_theta_lon_eq_mot_snd m g thetae (Zu,Zw,Zwd,Zq,Zeta,Ztau) (Ue,Ve,We)) theta t =
diff_eq_n_order 0 (lst_eta_lon_eq_mot_snd (Zu,Zw,Zwd,Zq,Zeta,Ztau)) eta t

Definition VI.9: third longitudinal equation of motion:

$\vdash_{\text{def}}$ ∀ Mw Mwd Mq Meta Mtau Mu. **lst_u_lon_eq_mot_trd** (Mu,Mw,Mwd,Mq,Meta,Mtau) = [−Mu]
$\vdash_{\text{def}}$ ∀ Mu Mq Meta Mtau Mw Mwd. **lst_w_lon_eq_mot_trd** (Mu,Mw,Mwd,Mq,Meta,Mtau) = [Mw; Mwd]
$\vdash_{\text{def}}$ ∀ Mu Mw Mwd Meta Mtau Ix Iz Mq Iy. **lst_theta_lon_eq_mot_trd** (Mu,Mw,Mwd,Mq,Meta,Mtau) (Ix,Iy,Iz) = [Cx (and0); −Mq; Iy]
$\vdash_{\text{def}}$ ∀ Mu Mw Mwd Mq Mtau Meta. **lst_eta_lon_eq_mot_trd** (Mu,Mw,Mwd,Mq,Meta,Mtau) = [Meta]
$\vdash_{\text{def}}$ ∀ Ue Ve We v u w Ix Iy Iz theta Mu Mw Mwd Mq Meta Mtau eta t. **lon_eq_mot_trd**
(Mu,Mw,Mwd,Mq,Meta,Mtau) (Ue,Ve,We) (u,v,w) (Ix,Iy,Iz) theta eta t ⇔
diff_eq_n_order 0 (lst_u_lon_eq_mot_trd (Mu,Mw,Mwd,Mq,Meta,Mtau)) u t -
diff_eq_n_order 1 (lst_w_lon_eq_mot_trd (Mu,Mw,Mwd,Mq,Meta,Mtau)) w t +
diff_eq_n_order 2 (lst_theta_lon_eq_mot_trd (Mu,Mw,Mwd,Mq,Meta,Mtau) (Ix,Iy,Iz)) theta t =
diff_eq_n_order 0 (lst_eta_lon_eq_mot_trd (Mu,Mw,Mwd,Mq,Meta,Mtau)) eta t

Next, in order to find out the transfer functions of the UAV corresponding to its various inputs and outputs, we take the Laplace transform of the longitudinal equations of motion [Eq. (2)]:

$$\begin{aligned}(ms - X_u)u(s) - (X_{\dot{w}}ws + X_w)w(s) - ((X_q - mW_e)s\\ - mg\cos\theta_e)\theta(s) = X_\eta\eta(s)\\ -Z_u u(s) - ((Z_{\dot{w}} - m)s + Z_w)w(s) - ((Z_q + mU_e)s\\ + mg\sin\theta_e)\theta(s) = Z_\eta\eta(s)\\ M_u u(s) - (M_{\dot{w}}s + M_w)w(s) + (I_y s^2 - M_q s)\theta(s) = M_\eta\eta(s)\end{aligned} \tag{3}$$

To verify the implication relationship between the longitudinal equations of motion and their corresponding Laplace transforms, we formalize a generalized equation of motion for an aircraft incorporating the multiple inputs and outputs as follows:

Definition VI.10: generic differential equation of aircraft (MIMO system):

$\vdash_{\text{def}}$ ∀ m fstlst w n sndlst x p trdlst y q fthlst z. **diff_eq_uav_gen** m n p q fstlst sndlst trdlst fthlst w x y z ⇔
(diff_eq_n_order m fstlst w t + diff_eq_n_order n sndlst x t + diff_eq_n_order p trdlst y t = diff_eq_n_order q fthlst z t)

We verify the Laplace transform of the preceding equation [i.e., Eq. (3)] as the following HOL Light theorem:

Theorem VI.1: Laplace transform of the differential equation of aircraft:

$\vdash_{\text{thm}}$ ∀ z y x w m n p q fstlst sndlst trdlst fthlst s.
Assumption A1: (∀ t. differen_higher_deriv m w t) ∧
Assumption A2: (∀ t. differen_higher_deriv n x t) ∧
Assumption A3: (∀ t. differen_higher_deriv p y t) ∧
Assumption A4: (∀ t. differen_higher_deriv q z t) ∧
Assumption A5: (0 <m ⇒zero_initial_conditions (m - 1) w) ∧
Assumption A6: (0 <n ⇒zero_initial_conditions (n - 1) w) ∧
Assumption A7: (0 <p ⇒zero_initial_conditions (p - 1) w) ∧
Assumption A8: (0 <q ⇒zero_initial_conditions (q - 1) w) ∧
Assumption A9: lap_exists_higher_der m w s ∧
Assumption A10: lap_exists_higher_der n x s ∧
Assumption A11: lap_exists_higher_der p y s ∧
Assumption A12: lap_exists_higher_der q z s ∧
Assumption A13: (∀ t. diff_eq_uav_gen m n p q fstlst sndlst trdlst fthlst w x y z
⇒lap_trans w s * vsum (0..m) (λk. EL k fstlst * s pow k) + lap_trans x s * vsum (0..n) (λk. EL k sndlst * s pow k) + lap_trans y s * vsum (0..p) (λk. EL k trdlst * s pow k) = lap_trans z s * vsum (0..q) (λk. EL k fthlst * s pow k)

where lap_trans presents the Laplace transform of a complex-valued function [48,49]. Assumptions A1–A4 provide the differentiability conditions for the higher-order derivatives of the input and outputs w, x, y, and z up to orders *m*, *n*, *p*, and *q*, respectively. Similarly, Assumptions A5–A8 present the zero initial conditions for the functions w, x, y, and z, respectively. Assumptions A9–A12 ensure that the Laplace transform of the functions w, x, y, and z exist up to orders *m*, *n*, *p*, and *q*, respectively. The last assumption (Assumption A13) provides the differential equation, modeling a generic equation of motion of an aircraft acting as a MIMO system. Finally, the conclusion represents the corresponding Laplace transform. The verification of the preceding theorem is mainly based on linearity of the Laplace existence, linearity of the Laplace transform, and the Laplace transform of a *n*-order differential equation properties.

Now, we verify the Laplace transform of the longitudinal equations of motion of the UAV as the following HOL Light theorem:

Lemma VI.1: Laplace transform of the longitudinal equations of motion:

$\vdash_{\text{thm}}$ ∀ s u v w theta eta Ue Ve We Zq Ztau Zu Zu Zw Zwd Zeta g m p q r thetae.
Assumption A1: (∀ t. differen_higher_deriv 1 u t) ∧
Assumption A2: (∀ t. differen_higher_deriv 1 w t) ∧
Assumption A3: (∀ t. differen_higher_deriv 2 theta t) ∧
Assumption A4: (∀ t. differen_higher_deriv 0 eta t) ∧
Assumption A5: zero_initial_conditions 0 u ∧
Assumption A6: zero_initial_conditions 0 w ∧
Assumption A7: zero_initial_conditions 1 theta ∧
Assumption A8: and0 <m ∧
Assumption A9: and0 <g ∧
Assumption A10: lap_exists_higher_der 1 u s ∧
Assumption A11: lap_exists_higher_der 1 w s ∧
Assumption A12: lap_exists_higher_der 2 theta s ∧
Assumption A13: lap_exists_higher_der 0 eta s ∧
Assumption A14: (∀ t. lon_eq_mot_fst (Xu,Xv,Xw,Xwd,Xq,Xeta,Xtau) (Ue,Ve,We) m g thetae (p,q,r) (u,v,w) theta eta t) ∧
Assumption A15: (∀ t. lon_eq_mot_snd (Zu,Zw,Zwd,Zq,Zeta,Ztau) (Ue,Ve,We) m g thetae (p,q,r) (u,v,w) theta eta t) ∧
Assumption A16: (∀ t. lon_eq_mot_trd (Mu,Mw,Mwd,Mq,Meta,Mtau) (Ue,Ve,We) (u,v,w) (Ix,Iy,Iz) theta eta t)
⇒lap_tran_lon_eq_of_mot_fst (Xu,Xv,Xw,Xwd,Xq,Xeta,Xtau) (Ue,Ve,We) m g thetae (u,v,w) theta eta s ∧

lap_tran_lon_eq_of_mot_snd (Zu,Zw,Zwd,Zq, Zeta,Ztau) (Ue,Ve,We) m g thetae (u,v,w) theta eta s ∧

lap_tran_lon_eq_of_mot_trd (Mu,Mw,Mwd,Mq, Meta,Mtau) (u,v,w) (Ix,Iy,Iz) theta eta s

Assumptions A1–A4 provide the conditions for the differentiability of the higher-order derivatives of the input and outputs u, w, theta, and eta up to orders 1, 1, 2, and 0, respectively. Similarly, Assumptions A5–A7 present the zero initial conditions for the functions u, w, and theta, respectively. Assumptions A8–A9 model the condition that the mass of the aircraft m and acceleration due to gravity g are positive. Similarly, Assumptions A10–A13 ensure that the Laplace transform of the functions u, w, theta, and eta exist up to orders 1, 1, 2, and 0, respectively. The last three assumptions (Assumptions A14–A16) provide the longitudinal equations of motion of the UAV. Finally, the conclusion represents the corresponding Laplace transform [Eq. (3)]. The formal reasoning of the preceding theorem is mainly based on Lemma VI.1 along with some complex arithmetic reasoning.

Next, we write the Laplace transform of the longitudinal equations of motion, i.e., Eq. (3) in matrix form:

$$\begin{bmatrix} (ms-X_u) & -(X_{\dot{w}}ws+X_w) & -((X_q-mW_e)s-mg\cos\theta_e) \\ -Z_u & -((Z_{\dot{w}}-m)s+Z_w) & -((Z_q+mU_e)s+mg\sin\theta_e) \\ M_u & -(M_{\dot{w}}s+M_w) & (I_y s^2-M_q s) \end{bmatrix} \times \begin{bmatrix} u(s) \\ w(s) \\ \theta(s) \end{bmatrix} = \begin{bmatrix} X_\eta \eta(s) \\ Z_\eta \eta(s) \\ M_\eta \eta(s) \end{bmatrix} \tag{4}$$

Now, by using Cramer's rule, the transfer functions of the UAV for various inputs and outputs corresponding to the longitudinal equations of motion are mathematically expressed as follows:

$$\frac{u(s)}{\eta(s)} \equiv \frac{N_\eta^u(s)}{\Delta_{\mathrm{lon}}(s)} \frac{w(s)}{\eta(s)} \equiv \frac{N_\eta^w(s)}{\Delta_{\mathrm{lon}}(s)} \frac{\theta(s)}{\eta(s)} \equiv \frac{N_\eta^\theta(s)}{\Delta_{\mathrm{lon}}(s)} \tag{5}$$

where

$$N_\eta^u(s) = \begin{vmatrix} X_\eta & -(X_{\dot{w}}ws+X_w) & -((X_q-mW_e)s-mg\cos\theta_e) \\ Z_\eta & -((Z_{\dot{w}}-m)s+Z_w) & -((Z_q+mU_e)s+mg\sin\theta_e) \\ M_\eta & -(M_{\dot{w}}s+M_w) & (I_y s^2-M_q s) \end{vmatrix}$$

$$N_\eta^w(s) = \begin{vmatrix} (ms-X_u) & X_\eta & -((X_q-mW_e)s-mg\cos\theta_e) \\ -Z_u & Z_\eta & -((Z_q+mU_e)s+mg\sin\theta_e) \\ M_u & M_\eta & (I_y s^2-M_q s) \end{vmatrix}$$

$$N_\eta^\theta(s) = \begin{vmatrix} (ms-X_u) & -(X_{\dot{w}}ws+X_w) & X_\eta \\ -Z_u & -((Z_{\dot{w}}-m)s+Z_w) & Z_\eta \\ M_u & -(M_{\dot{w}}s+M_w) & M_\eta \end{vmatrix}$$

$$\Delta_{\mathrm{lon}}(s) = \begin{vmatrix} (ms-X_u) & -(X_{\dot{w}}ws+X_w) & -((X_q-mW_e)s-mg\cos\theta_e) \\ -Z_u & -((Z_{\dot{w}}-m)s+Z_w) & -((Z_q+mU_e)s+mg\sin\theta_e) \\ M_u & -(M_{\dot{w}}s+M_w) & (I_y s^2-M_q s) \end{vmatrix}$$

We verify the transfer function for the input $\eta(t)$ and output $u(t)$, i.e., $\eta(s)/u(s)$ as the following HOL Light theorem:

Theorem VI.2: transfer function for the input $\eta(t)$ and output $u(t)$:

$\vdash_{\mathrm{thm}}$ ∀ Ix Iy Iz Meta Mq Mtau Mu Mw Mwd Ue Ve We Xeta Xq Xtau Xu Xv Xw Xwd Zeta Zq Ztau Zu Zw Zwd eta g m s theta thetae u v w.

Assumption A1: (∀t. differen_higher_deriv 1 u t) ∧
Assumption A2: (∀t. differen_higher_deriv 1 w t) ∧
Assumption A3: (∀t. differen_higher_deriv 2 theta t) ∧
Assumption A4: (∀ t. differen_higher_deriv 0 eta t) ∧
Assumption A5: zero_initial_conditions 0 u ∧
Assumption A6: zero_initial_conditions 0 w ∧
Assumption A7: zero_initial_conditions 1 theta ∧
Assumption A8: and0 <m ∧
Assumption A9: and0 <g ∧
Assumption A10: lap_exists_higher_der 1 u s ∧
Assumption A11: lap_exists_higher_der 1 w s ∧
Assumption A12: lap_exists_higher_der 2 theta s ∧
Assumption A13: lap_exists_higher_der 0 eta s ∧
Assumption A14: non_zero_denom_cond (Xu,Xv,Xw, Xwd,Xq,Xeta,Xtau) (Zu,Zw,Zwd,Zq,Zeta,Ztau) (Mu,Mw,Mwd,Mq,Meta,Mtau) (Ue,Ve,We) m g thetae (u,v,w) (Ix,Iy,Iz) theta eta s ∧
Assumption A15: (∀ t. lon_eq_mot_fst (Xu,Xv,Xw, Xwd,Xq,Xeta,Xtau) (Ue,Ve,We) m g thetae (p,q,r) (u,v,w) theta eta t) ∧
Assumption A16: (∀t. lon_eq_mot_snd (Zu,Zw,Zwd,Zq, Zeta,Ztau) (Ue,Ve,We) m g thetae (p,q,r) (u,v,w) theta eta t) ∧
Assumption A17: (∀t. lon_eq_mot_trd (Mu,Mw,Mwd,Mq, Meta,Mtau) (Ue,Ve,We) (u,v,w) (Ix,Iy,Iz) theta eta t)

⇒lap_trans u s / lap_trans eta s = cdet (lon_numer_poly_matrix_ueta (Xu,Xv,Xw,Xwd,Xq,Xeta, Xtau)
(Zu,Zw,Zwd,Zq,Zeta,Ztau) (Mu,Mw,Mwd,Mq,Meta, Mtau) (Ue,Ve,We) m g thetae (u,v,w) (Ix,Iy,Iz) theta eta s) /
cdet (lt_lon_eq_matrix (Xu,Xv,Xw,Xwd,Xq, Xeta,Xtau) (Zu,Zw,Zwd,Zq,Zeta,Ztau)
(Mu,Mw,Mwd,Mq,Meta,Mtau) (Ue,Ve,We) m g thetae (u,v,w) (Ix,Iy,Iz) theta eta s)

where cdet models the determinant of a complex-valued square matrix in HOL Light. Assumptions A1–A13 are the same as that of Lemma VI.1. Assumption A14 ensures that the denominators of the transfer function expression are nonzero. The last three assumptions (Assumptions A15–A17) provide the longitudinal equations of motion of the UAV. Finally, the conclusion represents the transfer function $u(s)/\eta(s)$. The proof process of the preceding theorem is mainly based on Lemma VI.1, the properties of vectors, and the complex matrices along with the following important lemma regarding application of Cramer's rule.

Lemma VI.2: application of Cramer's rule on a matrix representation:

$\vdash_{\mathrm{thm}}$ ∀ a b c d e f g h i u w z n p q r.

Assumption A1:

$$\begin{bmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} \\ \mathtt{d} & \mathtt{e} & \mathtt{f} \\ \mathtt{g} & \mathtt{h} & \mathtt{i} \end{bmatrix} ** \begin{bmatrix} \frac{\mathtt{u}}{\mathtt{n}} \\ \frac{\mathtt{w}}{\mathtt{n}} \\ \frac{\mathtt{z}}{\mathtt{n}} \end{bmatrix} = \begin{bmatrix} \mathtt{p} \\ \mathtt{q} \\ \mathtt{r} \end{bmatrix} \wedge$$

Assumption A2: (n ≠ Cx (and0)) ∧
Assumption A3: (a * d ≠ Cx (and0)) ∧
Assumption A4: (a * g ≠ Cx (and0)) ∧
Assumption A5: (d * g ≠ Cx (and0)) ∧

Assumption A6: (b * d * g - a * d * h ≠ Cx (and0)) ∧
Assumption A7: (b * d * g - a * e * g ≠ Cx (and0)) ∧
Assumption A8: (a * e * i - a * f * h - b * d * i + b * f * g + c * d * h - c * e * g = Cx (and0))
Assumption A9: ((b * d * g - a * d * h) * (c * d * g - a * f * g) - (b * d * g - a * e * g) * (c * d * g - a * d * i) ≠ Cx (and0)) ∧

$$\Rightarrow \frac{\mathtt{u}}{\mathtt{n}} = \frac{\mathtt{cdet}\begin{bmatrix} \mathtt{p} & \mathtt{b} & \mathtt{c} \\ \mathtt{q} & \mathtt{e} & \mathtt{f} \\ \mathtt{r} & \mathtt{h} & \mathtt{i} \end{bmatrix}}{\mathtt{cdet}\begin{bmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} \\ \mathtt{d} & \mathtt{e} & \mathtt{f} \\ \mathtt{g} & \mathtt{h} & \mathtt{i} \end{bmatrix}}$$

Assumption A1 presents the matrix representation of the equations modeling the continuous dynamics of a system. Assumptions A2–A9 ensure that the corresponding denominators are nonzero in the arithmetic manipulation of the matrix representation. Finally, the conclusion represents the transfer function $u(s)/n(s)$ by Cramer's rule. The verification of Lemma VI.2 is based on the properties of vectors and the complex matrices along with some complex arithmetic reasoning. We also formally verified the transfer functions w(s)/eta(s) and theta(s)/eta(s); and the details about their verification can be found in Ref. [47].

Next, in order to verify the longitudinal equations of motion of the UAV based on its Laplace transform, we verify an alternate representation of Lerch's theorem [48], which captures the dynamics of MIMO systems and is verified as the following theorem in HOL Light.

Theorem VI.3: alternate representation of Lerch's theorem:

$\vdash_{\text{thm}}$ ∀ z y x w m n p q fstlst sndlst trdlst fthlst r.
Assumption A1: (∀ t. differen_higher_deriv m w t) ∧
Assumption A2: (∀ t. differen_higher_deriv n x t) ∧
Assumption A3: (∀ t. differen_higher_deriv p y t) ∧
Assumption A4: (∀ t. differen_higher_deriv q z t) ∧
Assumption A5: (0 <m ⇒zero_initial_conditions (m - 1) w) ∧
Assumption A6: (0 <n ⇒zero_initial_conditions (n - 1) x) ∧
Assumption A7: (0 <p ⇒zero_initial_conditions (p - 1) y) ∧
Assumption A8: (0 <q ⇒zero_initial_conditions (q - 1) z) ∧
Assumption A9: and0 <Re r ∧
Assumption A10: (∀ s. Re r ≤Re s ⇒lap_exists_higher_der m w s) ∧
Assumption A11: (∀ s. Re r ≤Re s ⇒lap_exists_higher_der n x s) ∧
Assumption A12: (∀ s. Re r ≤Re s ⇒lap_exists_higher_der p y s) ∧
Assumption A13: (∀ s. Re r ≤Re s ⇒lap_exists_higher_der q z s) ∧
Assumption A14: (∀ s. Re r ≤Re s ⇒
lap_trans w s * vsum (0..m) (λk. EL k fstlst * s pow k) + lap_trans x s * vsum (0..n) (λk. EL k sndlst * s pow k) +
lap_trans y s * vsum (0..p) (λk. EL k trdlst * s pow k) = lap_trans z s * vsum (0..q) (λk. EL k fthlst * s pow k))
⇒(∀ t. and0 ≤*t*
⇒diff_eq_n_order m fstlst w t + diff_eq_n_order n sndlst x t + diff_eq_n_order p trdlst y t = diff_eq_n_order q fthlst z t)

where *t* converts a four-dimensional vector R^1 into a real number R. Assumptions A1–A8 are the same as that of Theorem VI.1. Assumption A9 ensures that the real part of the Laplace variable r is always positive. Assumptions A10–A13 ensure that the Laplace transforms of the functions w, x, y, and z exist up to orders m, n, p, and q, respectively. *Assumption A14* provides the Laplace transform of the equation, modeling the continuous dynamics of the MIMO system. Finally, the conclusion provides the continuous dynamics itself. The verification of Theorem VI.3 is based on Lerch's theorem [48], the linearity properties of the Laplace transform, and its existence.

Now, we use the preceding theorem to formally verify the longitudinal equations of motion of the UAV as the following HOL Light theorem:

Theorem VI.4: Laplace transform implies longitudinal equations of motion:

$\vdash_{\text{thm}}$ ∀ Ix Iy Iz Meta Mq Mtau Mu Mw Mwd Ue Ve We Xeta Xq Xtau Xu Xv Xw Xwd Zeta Zq Ztau Zu Zw Zwd eta g m s theta thetae u v w p q r r'.
Assumption A1: (∀ t. differen_higher_deriv 1 u t) ∧
Assumption A2: (∀ t. differen_higher_deriv 1 w t) ∧
Assumption A3: (∀ t. differen_higher_deriv 2 theta t) ∧
Assumption A4: (∀ t. differen_higher_deriv 0 eta t) ∧
Assumption A5: zero_initial_conditions 0 u ∧
Assumption A6: zero_initial_conditions 0 w ∧
Assumption A7: zero_initial_conditions 1 theta ∧
Assumption A8: and0 <m ∧
Assumption A9: and0 <g ∧
Assumption A10: and0 <Re r ∧
Assumption A11: (∀ s. Re r ≤Re s ⇒lap_exists_higher_der 1 u s) ∧
Assumption A12: (∀ s. Re r ≤Re s ⇒lap_exists_higher_der 1 w s) ∧
Assumption A13: (∀ s. Re r ≤Re s ⇒lap_exists_higher_der 2 theta s) ∧
Assumption A14: (∀ s. Re r ≤Re s ⇒lap_exists_higher_der 0 eta s) ∧
Assumption A15: (∀ s. Re r ≤Re s ⇒lap_tran_lon_eq_of_mot_fst (Xu,Xv,Xw,Xwd,Xq,Xeta,Xtau) (Ue,Ve,We) m g thetae (u,v,w) theta eta s) ∧
Assumption A16: (∀ s. Re r ≤Re s ⇒lap_tran_lon_eq_of_mot_snd (Zu,Zw,Zwd,Zq,Zeta,Ztau) (Ue,Ve,We) m g thetae (u,v,w) theta eta s) ∧
Assumption A17: (∀ s. Re r ≤Re s ⇒lap_tran_lon_eq_of_mot_trd (Mu,Mw,Mwd,Mq,Meta,Mtau) (u,v,w) (Ix,Iy,Iz) theta eta s)
⇒(∀ t. and0 ≤*t* ⇒lon_eq_mot_fst (Xu,Xv,Xw,Xwd,Xq,Xeta,Xtau) (Ue,Ve,We) m g thetae (p,q,r') (u,v,w) theta eta t) ∧
(∀ t. and0 ≤*t* ⇒lon_eq_mot_snd (Zu,Zw,Zwd,Zq,Zeta,Ztau) (Ue,Ve,We) m g thetae (p,q,r') (u,v,w) theta eta t) ∧
(∀ t. and0 ≤*t* ⇒lon_eq_mot_trd (Mu,Mw,Mwd,Mq,Meta,Mtau) (Ue,Ve,We) (u,v,w) (Ix,Iy,Iz) theta eta t)

Assumptions A1–A9 are the same as that of Theorem VI.2. Assumption A10 ensures that the real part of the Laplace variable r is always positive. Assumptions A11–A14 ensure that the Laplace transform of the functions u, w, theta, and eta exist up to orders 1, 1, 2, and 0, respectively. Assumptions A15–A17 provide the Laplace transform of the longitudinal equations of motion of the UAV. Finally, the conclusion provides the longitudinal equations of motion. The verification of the preceding theorem is based on the straightforward utility of Theorem VI.3 along with some complex arithmetic reasoning. This concludes our formalization of the longitudinal equations of motion of UAVs, their time- and frequency-domain solutions, and their associated transfer functions. The details about the formalization can be found in Ref. [47].

The lateral–directional motion of UAVs is described by the side force Y, rolling moment L, and yawing moment N. To formalize the lateral–directional equations of motion, we first model the aerodynamics stability derivatives corresponding to lateral–directional motion, represented in Table 2, using the type abbreviation feature of HOL Light.

Definition VI.11: aerodynamics stability derivatives:

new_type_abbrev ("aero_sta_deriv_lateral_force", `:(Yv × Yp × Yr × Yxi)`)

new_type_abbrev ("aero_sta_deriv_roll_moment", `:(Lv × Lp × Lr × Lxi)`)

new_type_abbrev ("aero_sta_deriv_yaw_moment", `:(Nv × Np × Nr × Nxi)`)

The lateral–directional equations of motion of the UAV are mathematical expressed by the following set of differential equations [8,10]:

$$\begin{aligned} & m\dot{v}-Y_v v-(Y_p+mW_e)p-(Y_r-mU_e)r-mg\phi\cos\theta_e-mg\psi\sin\theta_e \\ & \quad =Y_\xi\xi+Y_\zeta\zeta \\ & \quad -L_v v+I_x\dot{p}-L_p p-I_{xz}\dot{r}-L_r r=L_\xi\xi+L_\zeta\zeta \\ & \quad -N_v v-I_{xz}\dot{p}-N_p p+I_z\dot{r}-N_r r=N_\xi\xi+N_\zeta\zeta \end{aligned} \tag{6}$$

where $\dot{v}$, $\dot{p}$, and $\dot{r}$ represent the first-order derivatives of the linear and angular disturbance velocities, respectively. Similarly, the variables ξ and ζ model the aileron and rudder angles, respectively. Under the conditions $p(t)=\dot{\phi}(t)$, $r(t)=\dot{\psi}(t)$, and $\zeta(t)=0$, the preceding lateral–directional equations of motion become [10]

$$\begin{aligned} & m\dot{v}-Y_v v-(Y_p+mW_e)\dot{\phi}-(Y_r-mU_e)\dot{\psi}-mg\phi\cos\theta_e \\ & \quad -mg\psi\sin\theta_e=Y_\xi\xi \\ & \quad -L_v v+I_x\ddot{\phi}-L_p\dot{\phi}-I_{xz}\ddot{\psi}-L_r\dot{\psi}=L_\xi\xi \\ & \quad -N_v v-I_{xz}\ddot{\phi}-N_p\dot{\phi}+I_z\ddot{\psi}-N_r\dot{\psi}=N_\xi\xi \end{aligned} \tag{7}$$

We formalize the first differential equation of the lateral–directional equations of motion of the UAV [Eq. (7)] as follows:

Definition VI.12: First lateral–directional equation of motion:

$\vdash_{def}$ ∀ Yp Yr Yxi Yv m. **lst_v_lat_eq_mot_fst** m (Yv, Yp, Yr, Yxi) = [-Yv; Cx m]

Table 2 Aerodynamics stability derivatives and their data types

Symbol	Type
Yv	ℂ
Yp	ℂ
Yr	ℂ
Yxi	ℂ
Lv	ℂ
Lr	ℂ
Lp	ℂ
Lxi	ℂ
Nv	ℂ
Nr	ℂ
Np	ℂ
Nxi	ℂ

$\vdash_{def}$ ∀ Ue Ve Yv Yr Yxi g thetae Yp m We. **lst_phi_lat_eq_mot_fst** m g thetae (Ue,Ve,We) (Yv,Yp,Yr,Yxi)
= [Cx m * Cx g * ccos thetae; Yp + Cx m * We]

$\vdash_{def}$ ∀ Yv Yp Yxi Ve We g thetae Yr m Ue. **lst_psi_lat_eq_mot_fst** m g thetae (Yv,Yp,Yr,Yxi) (Ue,Ve,We)
= [Cx m * Cx g * csin thetae; Yr - Cx m * Ue]

$\vdash_{def}$ ∀ Yv Yp Yr Yxi. **lst_xi_lat_eq_mot_fst** (Yv,Yp,Yr,Yxi) = [Yxi]

$\vdash_{def}$ ∀ u w v phi m g thetae Ue Ve We psi Yv Yp Yr Yxi xi t.
lat_eq_mot_fst (Yv,Yp,Yr,Yxi) (Ue,Ve,We) m g thetae (u,v,w) phi psi xi t ⇔
diff_eq_n_order 1 (lst_v_lat_eq_mot_fst m (Yv,Yp,Yr,Yxi)) v t -
diff_eq_n_order 1 (lst_phi_lat_eq_mot_fst m g thetae (Ue,Ve,We) (Yv,Yp,Yr,Yxi)) phi t -
diff_eq_n_order 1 (lst_psi_lat_eq_mot_fst m g thetae (Yv,Yp,Yr,Yxi) (Ue,Ve,We)) psi t =
diff_eq_n_order 0 (lst_xi_lat_eq_mot_fst (Yv,Yp,Yr,Yxi)) xi t

where the function lat_eq_mot_fst accepts the function variables v, phi, psi, and xi, as well as the lists of coefficients lst_v_lat_eq_mot_fst,lst_phi_lat_eq_mot_fst, lst_psi_lat_eq_mot_fst and lst_xi_lat_eq_mot_fst, and returns the corresponding differential equation. Similarly, we model the other two differential equations of the lateral–directional equations of motion of the UAV [Eq. (7)] as follows:

Definition VI.13: second lateral–directional equation of motion:

$\vdash_{def}$ ∀ Lp Lr Lxi Lv. **lst_v_lat_eq_mot_snd** (Lv,Lp,Lr,Lxi) = [-Lv]

$\vdash_{def}$ ∀ Iy Iz Lv Lr Lxi Lp Ix. **lst_phi_lat_eq_mot_snd**(Ix,Iy,Iz) (Lv,Lp,Lr,Lxi) = [Cx (and0); Lp; -Ix]

$\vdash_{def}$ ∀ Lv Lp Lxi Lr Ixz. **lst_psi_lat_eq_mot_snd** Ixz (Lv,Lp,Lr,Lxi) = [Cx (and0); Lr; Ixz]

$\vdash_{def}$ ∀ Lv Lp Lr Lxi. **lst_xi_lat_eq_mot_snd** (Lv,Lp,Lr,Lxi) = [Lxi]

$\vdash_{def}$ ∀ u w v Ix Iy Iz phi Ixz psi Lv Lp Lr Lxi xi t. **lat_eq_mot_snd** (Lv,Lp,Lr,Lxi) Ixz (Ix,Iy,Iz) (u,v,w) phi psi xi t ⇔
diff_eq_n_order 0 (lst_v_lat_eq_mot_snd (Lv,Lp,Lr,Lxi)) v t -
diff_eq_n_order 2 (lst_phi_lat_eq_mot_snd (Ix,Iy,Iz) (Lv,Lp,Lr,Lxi)) phi t -
diff_eq_n_order 2 (lst_psi_lat_eq_mot_snd Ixz (Lv,Lp,Lr,Lxi)) psi t =
diff_eq_n_order 0 (lst_xi_lat_eq_mot_snd (Lv,Lp,Lr,Lxi)) xi t

Definition VI.14: third lateral–directional equation of motion:

$\vdash_{def}$ ∀ Np Nr Nxi Nv. **lst_v_lat_eq_mot_trd** (Nv,Np,Nr,Nxi) = [-Nv]

$\vdash_{def}$ ∀ Nv Nr Nxi Np Ixz. **lst_phi_lat_eq_mot_trd** Ixz (Nv,Np,Nr,Nxi) = [Cx (and0); Np; Ixz]

$\vdash_{def}$ ∀ Nv Np Nxi Ix Iy Nr Iz. **lst_psi_lat_eq_mot_trd** (Nv,Np,Nr,Nxi) (Ix,Iy,Iz) = [Cx (and0); Nr; -Iz]

$\vdash_{def}$ ∀ Nv Np Nr Nxi. **lst_xi_lat_eq_mot_trd** (Nv,Np,Nr,Nxi) = [Nxi]

```
⊢def ∀ u w v Ixz phi Ix Iy Iz psi Nv Np Nr Nxi xi t.
lat_eq_mot_trd (Nv,Np,Nr,Nxi) (Ix,Iy,Iz) Ixz
(u,v,w) phi psi xi t ⇔
     diff_eq_n_order 0 (lst_v_lat_eq_mot_trd
(Nv,Np,Nr,Nxi)) v t -
     diff_eq_n_order 2 (lst_phi_lat_eq_mot_trd
Ixz (Nv,Np,Nr,Nxi)) phi t -
     diff_eq_n_order 2 (lst_psi_lat_eq_mot_trd
(Nv,Np,Nr,Nxi) (Ix,Iy,Iz)) psi t =
     diff_eq_n_order 0 (lst_xi_lat_eq_mot_trd
(Nv,Np,Nr,Nxi)) xi t
```

Next, in order to find out the transfer functions of the UAV corresponding to its various inputs and outputs, we take the Laplace transform of the lateral–directional equations of motion [Eq. (7)].

$$\begin{aligned}
&(ms - Y_v)v(s) - ((Y_p + mW_e)s + mg\cos\theta_e)\phi(s) - ((Y_r - mV_e)s \\
&\quad + mg\sin\theta_e)\psi(s) = Y_\xi \xi(s) \\
&-L_v v(s) + (I_x s^2 - L_p s)\phi(s) - (I_{xz} + L_r s)\psi(s) = L_\xi \xi(s) \\
&-N_v v(s) - (I_{xz} s^2 + N_p s)\phi(s) - (I_z - N_r s)\psi(s) = L_\xi \xi(s) \quad (8)
\end{aligned}$$

We also formally verified the Laplace transform of the lateral–directional equations of motion, i.e., Eq. (8) and various transfer functions, such as $v(s)/\xi(s)$, $\Delta(s)/\xi(s)$, and $\theta(s)/\xi(s)$. The details about their verification can be found in Ref. [47].

VII. Formal Stability Analysis of CropCam UAV

CropCam UAV [50,51] is a low-altitude flying autopilot aircraft that provides the Global Positioning System (GPS)-based digital images and is widely used for remote sensing in various applications, such as large-scale topographic mapping, georeferencing, agriculture, etc. It is equipped with an autopilot, a GPS, and a digital camera; and it can provide high-resolution images, which are sometime difficult to obtain using traditional means, such as the satellite and a manned aircraft. Stability is an important control characteristic of UAVs that dampens out any oscillation in the aircraft motion caused by various disturbances, and thus restores UAVs to the equilibrium flight conditions [52]. Thus, a stable UAV provides a stable response (output) to a bounded input. It depends on the transfer functions of the UAV that are obtained as a result of analyzing the continuous dynamics of the aircraft modeled as the longitudinal and lateral–directional equations of motion.

Generally, the transfer function of a system is mathematically expressed as

$$\frac{Y(s)}{X(s)} = \frac{\mathrm{Num}(s)}{\mathrm{Denom}(s)} = \frac{b_p s^p + b_{p-1} s^{p-1} + \ldots + b_0}{a_q s^q + a_{q-1} s^{q-1} + \ldots + a_0} \qquad (9)$$

where $X(s)$ and $Y(s)$ present the Laplace transform of the input function $x(t)$ and output function $y(t)$, respectively. Similarly, Num(s) and Denom(s) are complex-valued polynomials. The equation Denom$(s) = 0$ is known as the characteristic equation, and its roots are called the poles of the system. The locations of these poles in the complex plane provide important information about the stability of the corresponding system. A system is said to be stable if all the poles are located on the left half of the complex plane [52].

We model the notion of the stability of a UAV as the following HOL Light function:

Definition VII.1: stability of a UAV:

```
⊢def ∀. is_stable_uav F = {s |F s = Cx (and0) ∧ Re s
<and0} ≠ EMPTY
```

where `is_stable_uav` accepts the denominator of the transfer function corresponding to the longitudinal and lateral–directional equations of motion of a UAV (i.e., `F`: $C \to C$) and provides a stable UAV. Similarly, `s`: C represents the root of the characteristic equation. The conjunct `F s = Cx (and0)` provides the characteristic equation. Similarly, `Re s <and0` models the condition that the poles of the UAV lie in the left half-complex plane.

The longitudinal equations of motion for the CropCam UAV are mathematically expressed in the matrix form as

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} z_u & z_w & 0 \\ m_u & m_w & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ \theta \end{bmatrix} = \begin{bmatrix} z_\eta \\ m_\eta \\ 0 \end{bmatrix} \eta \qquad (10)$$

Alternatively, we can write Eq. (10) as follows:

$$\begin{bmatrix} \dot{u} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} z_u & z_w \\ m_u & m_w \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} z_\eta \\ m_\eta \end{bmatrix} \eta \qquad (11)$$

Next, in order to find out the transfer functions of the CropCam UAV corresponding to its various inputs and outputs, we take the Laplace transform of preceding equation:

$$\begin{bmatrix} s - z_u & -z_w \\ s - m_u & -m_w \end{bmatrix} \begin{bmatrix} u(s) \\ w(s) \end{bmatrix} = \begin{bmatrix} z_\eta \eta(s) \\ m_\eta \eta(s) \end{bmatrix} \qquad (12)$$

The transfer function $u(s)/\eta(s)$ corresponding to the longitudinal equations of motion for the CropCam UAV [Eq. (11)] is mathematically expressed as [8]

$$\frac{u(s)}{\eta(s)} = \frac{X_\eta[s + Z_w + X_w(Z_\eta/X_\eta)]}{s^2 - (Z_w + X_u)s + (Z_w X_u - Z_u X_w)} \qquad (13)$$

We verify the transfer function $u(s)/\eta(s)$ [Eq. (13)] as the following HOL Light theorem:

Theorem VII.1: transfer function for the input $\eta(t)$ and output $u(t)$:

```
⊢thm ∀ Ix Iy Iz Meta Mq Mtau Mu Mw Mwd Ue Ve We Xeta Xq
Xtau Xu Xv Xw Xwd Zeta Zq Ztau Zu Zw Zwd s u v w.
     Assumption A1: (∀ t. differen_higher_deriv 2 u t) ∧
     Assumption A2: (∀ t. differen_higher_deriv 1 w t) ∧
     Assumption A3: zero_initial_conditions 1 u ∧
     Assumption A4: zero_initial_conditions 0 w ∧
     Assumption A5: lap_exists_higher_der 2 u s ∧
     Assumption A6: lap_exists_higher_der 1 w s ∧
     Assumption A7: nzero_denom_cropcam (Xu,Xv,Xw,
Xwd,Xq,Xeta,Xtau) (Zu,Zw,Zwd,Zq,Zeta,Ztau) (u,
v,w) eta s ∧
     Assumption A8: (∀ t. lon_eq_mot_fst_cropcam (Xu,
Xv,Xw,Xwd,Xq,Xeta,Xtau) (u,v,w) eta t) ∧
     Assumption A9: (∀ t. lon_eq_mot_snd_cropcam (Zu,
Zw,Zwd,Zq,Zeta,Ztau) (u,v,w) eta t) ∧
   ⇒lap_trans u s / lap_trans eta s = cdet (lon_
numer_poly_matrix_ueta_cropcam (Xu,Xv,Xw,Xwd,
Xq,Xeta,Xtau)
       (Zu,Zw,Zwd,Zq,Zeta,Ztau) (u,v,w) eta s) /
         cdet (lt_lon_eq_matrix_cropcam (Xu,Xv,
Xw,Xwd,Xq,Xeta,Xtau) (Zu,Zw,Zwd,Zq,Zeta,Ztau)
(u,v,w) eta s)
```

Assumptions A1–A2 assert the conditions for the differentiability of the higher-order derivatives of the input and outputs `u` and `w` up to orders 2 and 1, respectively. Similarly, Assumptions A3–A4 provide the zero initial conditions for the functions `u` and `w`, respectively. Assumptions A5–A6 present the conditions that the Laplace transform of the functions `u` and `w` exist up to orders 2 and 1, respectively. Assumption A7 ensures that the denominators of the transfer function expression are nonzero. The last two assumptions (Assumptions A8–A9) provide the longitudinal equations of motion of the CropCam UAV. Finally, the conclusion represents the transfer function $u(s)/\eta(s)$. The proof process of the preceding theorem is very similar to that of Theorem VI.2.

Next, we use the formally verified transfer function $(u(s)/\eta(s))$ of the CropCam UAV for formally verifying its stability, requiring the denominator of the transfer function [Eq. (13)], as the following HOL Light theorem:

Theorem VII.2: stable CropCam UAV: $\vdash_{\text{thm}} \forall\ \alpha Z_w Z_u X_w X_u.$

Assumption A1: $((\mathtt{ReZ_w} + \mathtt{ReX_u} < \&\mathtt{0} \wedge$

$$((\mathtt{ReZ_w} + \mathtt{ReX_u})^2 - \&\mathtt{4} * (\mathtt{ReZ_w} * \mathtt{ReX_u} - \mathtt{ReZ_u} * \mathtt{ReX_w}) < \&\mathtt{0} \vee$$

$$(\mathtt{ReZ_w} + \mathtt{ReX_u})^2 - \&\mathtt{4} * (\mathtt{ReZ_w} * \mathtt{ReX_u} - \mathtt{ReZ_u} * \mathtt{ReX_w}) = \&\mathtt{0})) \vee$$

$$(\&\mathtt{0} < (\mathtt{ReZ_w} + \mathtt{ReX_u})^2 - \&\mathtt{4} * (\mathtt{ReZ_w} * \mathtt{ReX_u} - \mathtt{ReZ_u} * \mathtt{ReX_w}) \wedge$$

$$(\sqrt{(\mathtt{ReZ_w} + \mathtt{ReX_u})^2 - \&\mathtt{4} * (\mathtt{ReZ_w} * \mathtt{ReX_u} - \mathtt{ReZ_u} * \mathtt{ReX_w})} < -(\mathtt{ReZ_w} + \mathtt{ReX_u}) \vee$$

$$(\mathtt{ReZ_w} + \mathtt{ReX_u}) < \sqrt{(\mathtt{ReZ_w} + \mathtt{ReX_u})^2 - \&\mathtt{4} * (\mathtt{ReZ_w} * \mathtt{ReX_u} - \mathtt{ReZ_u} * \mathtt{ReX_w})}))) \wedge$$

Assumption A2: $\mathtt{Im\ Z_w} = \mathtt{and0} \wedge$
Assumption A3: $\mathtt{Im\ Z_u} = \mathtt{and0} \wedge$
Assumption A4: $\mathtt{Im\ X_w} = \mathtt{and0} \wedge$
Assumption A5: $\mathtt{Im\ X_u} = \mathtt{and0} \wedge$

$$\Rightarrow \mathtt{is\_stable\_uav}\ (\lambda \mathtt{s}.\ s^2 - s * (Z_w + X_u) + Z_w * X_u - Z_u * X_w)$$

Assumptions A1–A5 provide constraints for the stability of the CropCam UAV. The conclusion of the preceding theorem ensures that the CropCam UAV is stable. The verification of Theorem VII.2 is mainly based on Definition VII.1, along with some complex arithmetic reasoning. Similarly, we formally verify the transfer function and the stability of the CropCam UAV based on its transfer function $w(s)/\eta(s)$ corresponding to its longitudinal equations of motion. The details about the analysis can be found in the corresponding HOL Light proof script, which is available in Ref. [47].

The distinguishing features of our proposed framework as compared to the traditional analysis techniques are that all of the parameters along with their types, contributing to the continuous dynamics of UAVs, are clearly defined in our formal analysis. On the other hand, there is always a chance of misinterpreting any of these parameters in the dynamical analysis of UAVs using traditional techniques. Also, all of the verified theorems are of a generic nature; i.e., all of the functions and variables are universally quantified, and thus can be specialized for a particular scenario. However, in the case of computer-based simulations, we need to model each of the cases individually. Moreover, the inherent soundness of the theorem proving technique ensures that all the required assumptions are explicitly present along with the theorem. Similarly, the high expressiveness of the higher-order logic enables us to model the dynamics of UAVs (i.e., the differential equations-based equations of motion, the corresponding transfer function, and the frequency response in their true continuous form); whereas in the model checking-based analysis, they are mostly discretized and modeled using a state-transition system, which may compromise the accuracy of the analysis. The verification of the coordinate frames and their transformation as well as the formal analysis of the continuous dynamics of UAVs ensure the accurate orientation, position, and the motion of UAVs, and thus provide the safety of these aircraft. Similarly, the formal analysis of the CropCam UAV also ensures the stability of the aircraft corresponding to its various transfer functions.

VIII. Conclusions

UAVs are widely used in safety and mission-critical domains, such as rescue missions, transportation, surveillance, etc. Due to this fact, their accurate analysis is of utmost importance. In this paper, a framework was proposed for formally analyzing the dynamical aspects of UAVs using higher-order-logic theorem proving. First, various coordinate frames were formalized, such as navigation's and aircraft's body-fixed frames; and their associated transformation using HOL Light was formally verified. This requires the formalization of the complex matrices, which are also developed as a part of the proposed framework. The formalization of these coordinate frames ensures the correct orientation and position of the aircraft. The longitudinal and lateral–directional equations of motion for UAVs were also formalized, and their solutions were formally verified in the time and frequency domains. Finally, the stability analysis of the CropCam UAV was performed to ascertain the correct orientation and movement of these aircraft. In future, the aim is to extend the formalization of the coordinate frames to incorporate other coordinate systems such as quaternions.

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