Formalization of Asymptotic Notations in HOL4

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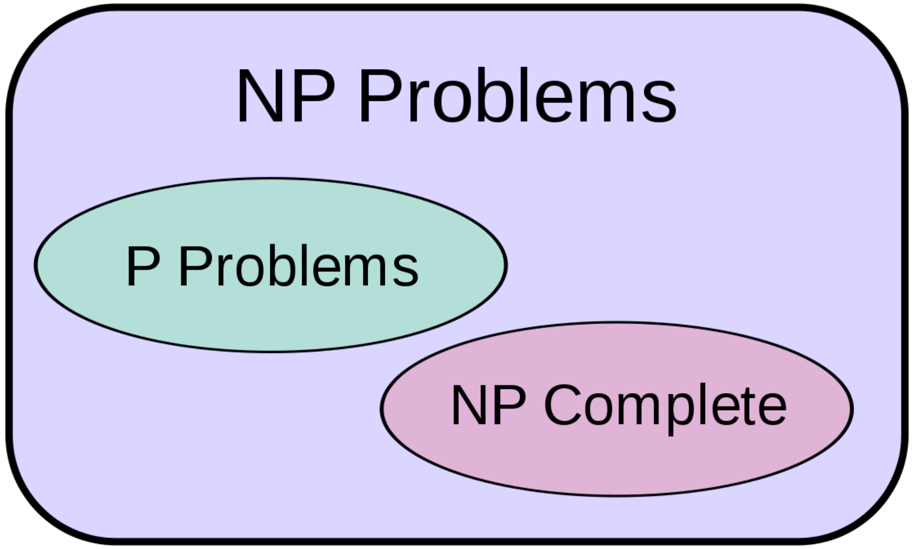
Outline

- Introduction
- Proposed Methodology
- Formalization Definitions
- Formal Verification
- Conclusions

Asymptotic Notations

- Used for computational time assessment of Algorithms

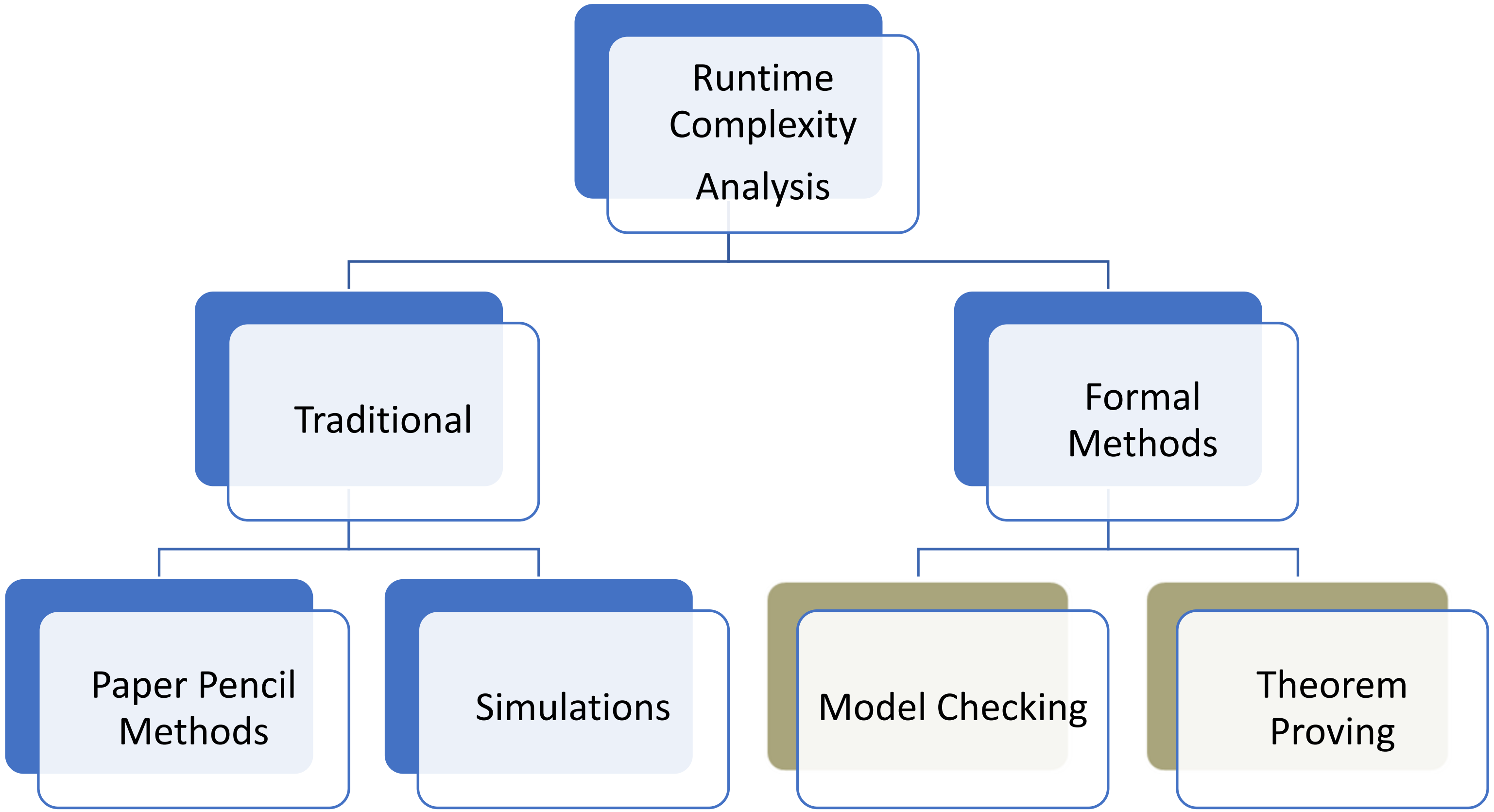
- The asymptotic notation based assessment is independent of
  - Language
  - Execution Platform
  - Compiler
  - Input data



Asymptotic Notations

- Big-O notation or simply O-notation was introduced by a number theorist Bachmann in 1894
- Little-o notation was introduced by Landau in 1909
- Big-Ω, Big-Θ, and Little-ω notations were presented by Knuth in 1976

Types of Analysis



Comparison of Analysis Techniques

Criteria	Paper-and-Pencil Proof	Simulation	Model Checking	Higher-order-logic Proof Assistants
Expressiveness	☑	☑	☒	☑
Accuracy	☒	☒	☑	☑
Automation	☒	☑	☑	☒

Given the extensive usage of asymptotic analysis of algorithms in safety-critical systems, there is a dire need of using formal methods support in this domain

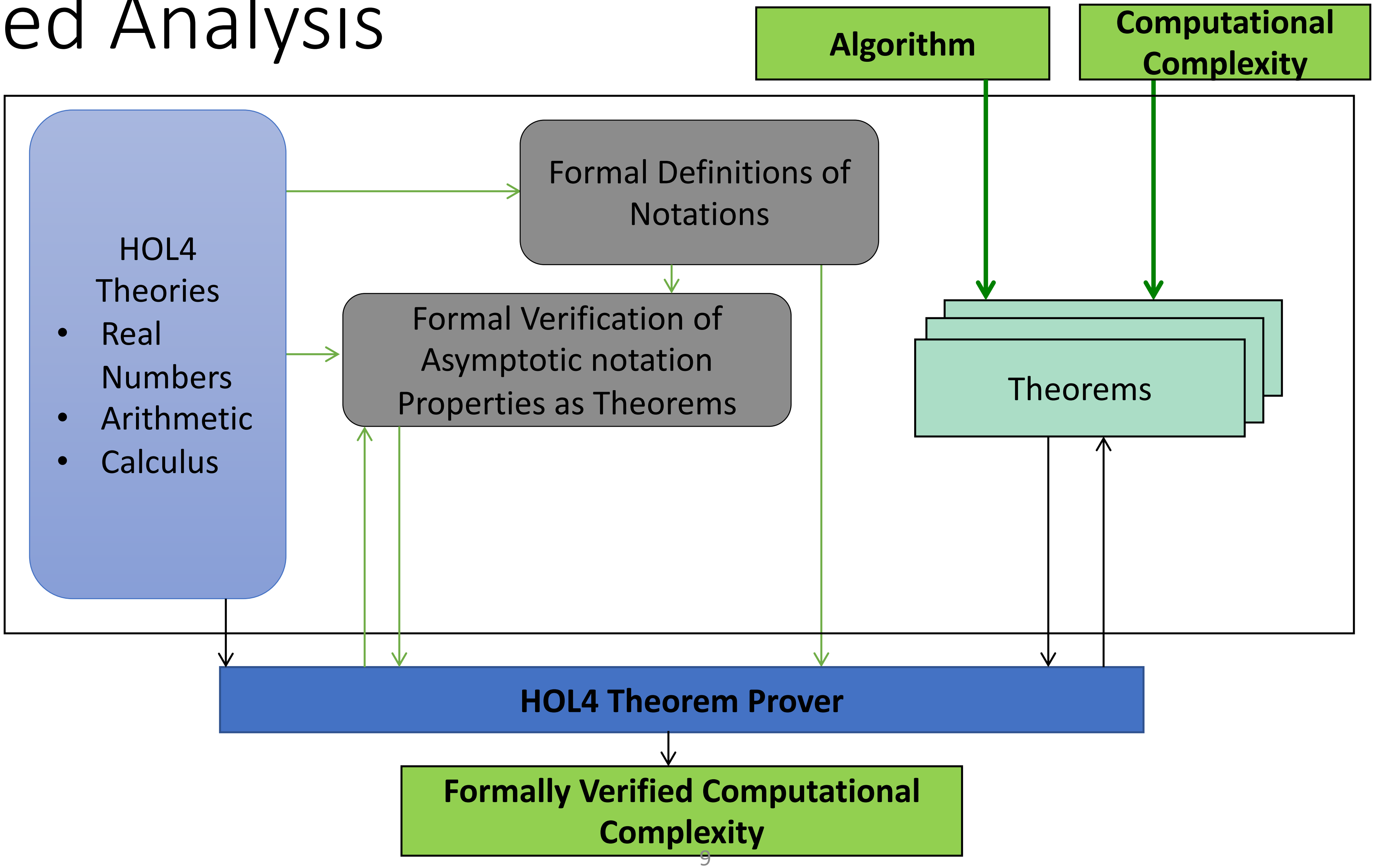
Contributions of this paper

- A library of formalized asymptotic notations, i.e., O, Θ, Ω, o and ω, in the higher-order-logic theorem prover HOL4 using the real number theory
  - Formal Definitions of O, Θ, Ω, o and ω *in HOL4*
  - Formal Verification of Properties of O, Θ, Ω, o and ω *in HOL4*

HOL4 Theorem Prover

- Higher-order-logic Proof Assistant
  - Notation: ML
  - Small Core:
    - 5 basic axioms
    - 8 primitive inference rules
- Numerous automatic proof procedures are available
- Supports Reasoning about
  - Real Numbers
  - Calculus

Proposed Approach for Asymptotic Notations Based Analysis



Formal Definitions: BigO

- The BigO takes a function *g* as an input and returns the set of all functions *f* which qualify the condition $0 \leq f(n) \leq c * g(n)$

Definition 1: *BigO Notation*

$$\vdash \forall\ g.\ \mathtt{BigO}\ (g:\mathtt{num} \rightarrow \mathtt{real}) = \{(f:\mathtt{num} \rightarrow \mathtt{real}) \mid (\exists\ c\ n\_0.(\forall\ n.\ n\_0 \leq n \land 0 < c \Longrightarrow 0 \leq f(n) \leq c\ g(n)))\}$$

- Here f and g are functions which take a natural number *num* and return a real number *real*
- The constants c and n_0 are of type real and num, respectively

Formal Definitions

Definition 2: *BigTheta Notation*

$$\vdash \forall\ g.\ \texttt{BigTheta}\ (g:num \rightarrow real) = \{(f:num \rightarrow real) \mid (\exists\ c1\ c2\ n\_0.(\forall\ n.n\_0 \leq n \wedge 0 < c1 \wedge 0 < c2 \Longrightarrow 0 \leq c1\ g(n) \leq f(n) \leq c2\ g(n)))\}$$

Definition 3: *BigOmega Notation*

$$\vdash \forall\ g.\ \texttt{BigOmega}\ (g:num \rightarrow real) = \{(f:num \rightarrow real) \mid (\exists\ c\ n\_0.(\forall\ n.\ n\_0 \leq n \wedge 0 < c \Longrightarrow 0 \leq c\ g(n) \leq f(n)))\}$$

Definition 4: *LittleO Notation*

$$\vdash \forall\ g.\ \texttt{LittleO}\ (g:num \rightarrow real) = \{(f:num \rightarrow real) \mid (\exists\ c\ n\_0.(\forall\ n.\ n\_0 \leq n \wedge 0 < c \Longrightarrow 0 \leq f(n) < c\ g(n)))\}$$

Definition 5: *LittleOmega Notation*

$$\vdash \forall\ g.\ \texttt{LittleOmega}\ (g:num \rightarrow real) = \{(f:num \rightarrow real) \mid (\exists\ c\ n\_0.(\forall\ n.\ n\_0 \leq n \wedge 0 < c \Longrightarrow 0 \leq c\ g(n) < f(n)))\}$$

Formal Verification

- Using the formal definitions of Asymptotic notations, we formally verified their properties
  - Transitivity
  - Symmetry
  - Transpose symmetry
  - Reflexivity
- using the HOL4 theorem prover

- The properties not only ensure the correctness of our definitions but also play a vital role in the formal complexity analysis of algorithms

Properties of O Notation

Theorem 1: *Transitivity of O-Notation*

$\vdash \forall$ `f g h. f` $\in$ `(BigO g)` $\wedge$ `g` $\in$ `(BigO h)` $\Longrightarrow$ `f` $\in$ `(BigO h)`

Theorem 2: *Sum of O-Notation*

$\vdash \forall$ `t1 t2 g1 g2. t1` $\in$ `(BigO g1)` $\wedge$ `t2` $\in$ `(BigO g2)` $\Longrightarrow$
`(t1 n + t2 n)` $\in$ `(BigO (max (g1 n, g2 n)))`

Conclusions

- Formalization of Asymtotic notations (O, Θ, Ω, o and ω) in HOL4
- Formal Framework for computational complexity analysis of algorithms

- Advantages
  - Accurate Results
  - Reduction in user-effort while formally Helpful in discovery of different pathways

Future Directions

- Applications in Cryptography
  - To estimate the size of the key so that it will be infeasible to break a system using given number of steps
  - Security assessment of authentication protocols, such as, the security proof of password authentication protocols

Thanks!

- Questions