Formal Kinematic Analysis of the Two-Link Planar Manipulator

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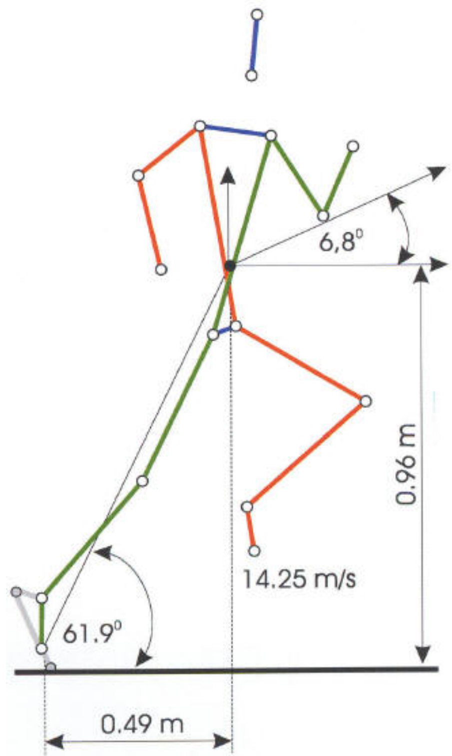
Outline

- Introduction
- Motivation
- Proposed Approach
- Case Study: A Biped Robot
- Conclusions

Kinematic Analysis

- Used to determine the operational range of a mechanical structure

Identify the links (rigid bodies) and joints (allow rotation or sliding)



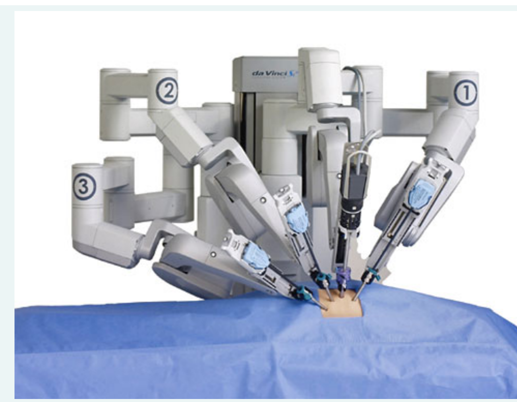
Kinematic Diagram

Determine the operational range using the principles of Geometry

Usage of Kinematic Analysis

- Allows us to determine the position, displacement, rotation, speed, velocity and acceleration of the given mechanical structure

- Integral step in the robotic design process to assess the workspace and precision

Slope climbing capability of a Biped Robot	Workspace of a Surgical Robot
	

Kinematic Analysis Approaches

- Paper-and-pencil proof methods
- Computer Simulations
  - Not recommended for Kinematic analysis of manipulators used in safety-critical applications
    - Y. Wang and G.S. Chirikjian. Propagation of Errors in Hybrid Manipulators. ICRA-06
  - Inaccuracies in kinematic analysis could lead to disastrous consequences, including a robot's breakdown
    - J.P. Merlet. A Formal Numerical Approach for Robust In-Workspace Singularity Detection. IEEE Transactions on Robotics, 2007

Formal Methods and Kinematic Analysis

- Computer Algebra Systems
  - *Kinematic and Dynamic Analysis in Mathematica*
  - Cannot be considered 100% reliable due to the involvement of un-verified symbolic manipulation algorithms in their core
- Automated Theorem Proving and State-based exhaustive approaches
  - Cannot guarantee absolute precision due to the continuous nature of the analysis
- **Proof Assistants using Higher-order logic**
  - Underlying principles of kinematic analysis have not been formalized in higher-order logic so far

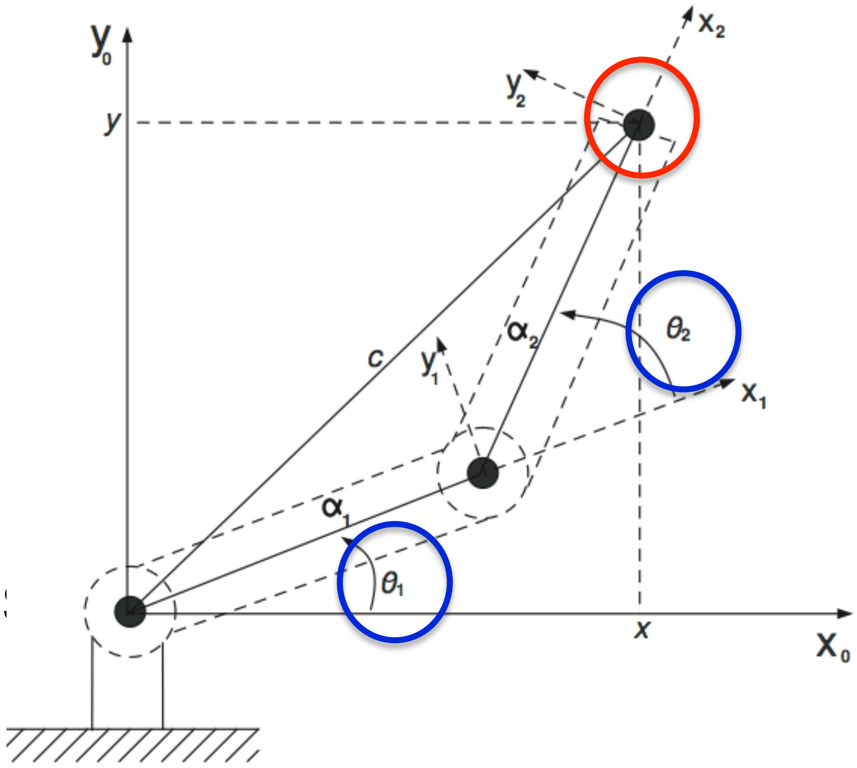
Our Contributions

- Formal Reasoning support for a Two-Link Planar manipulator in Higher-order logic
  - Formal verification of Forward and Inverse Kinematic relations for a Two-link Planar manipulator
- Built upon the Analytical Geometry Theory available in the HOL-Light Theorem prover
  - Supports n-dimensional Real Vectors and transcendental functions
- Case Study: Kinematic Analysis of a Biped Robot using the HOL-Light Theorem Prover

Two-Link Planar Manipulator

- A 2 Degrees of Freedom (DoF) System in the same plane
  - Requires two independent coordinates to describe its motion

- Relationship of the end-effectors tip with the angles θ_1 and θ_2

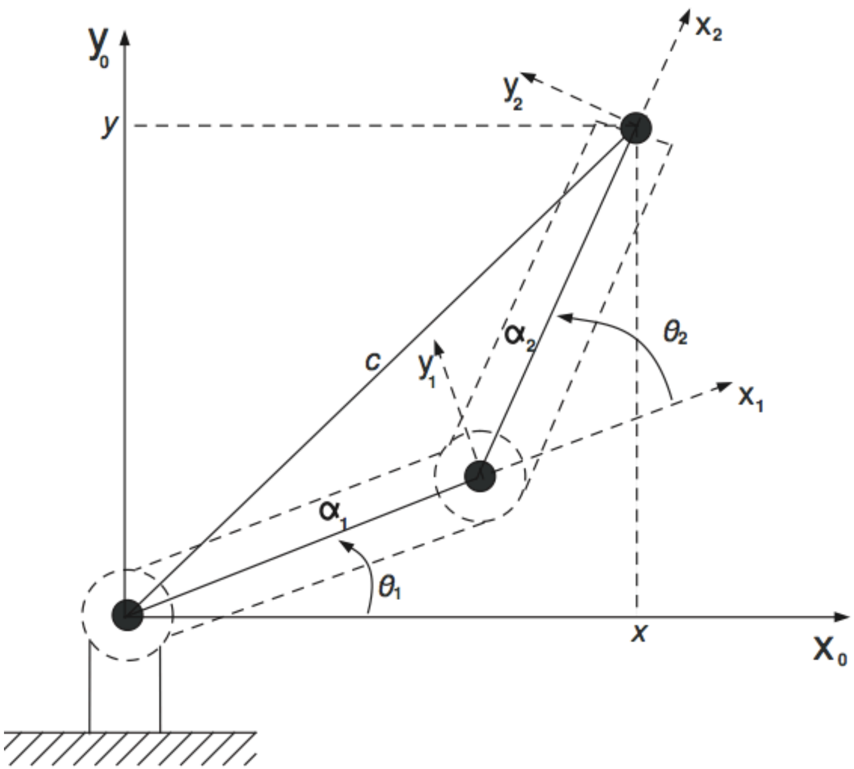


Forward Kinematics

- Cartesian position of the end-effector of the manipulator in terms of the joint angles

$$x = \alpha_1 \cos\theta_1 + \alpha_2 \cos(\theta_1 + \theta_2)$$

$$y = \alpha_1 \sin\theta_1 + \alpha_2 \sin(\theta_1 + \theta_2)$$



Inverse Kinematics

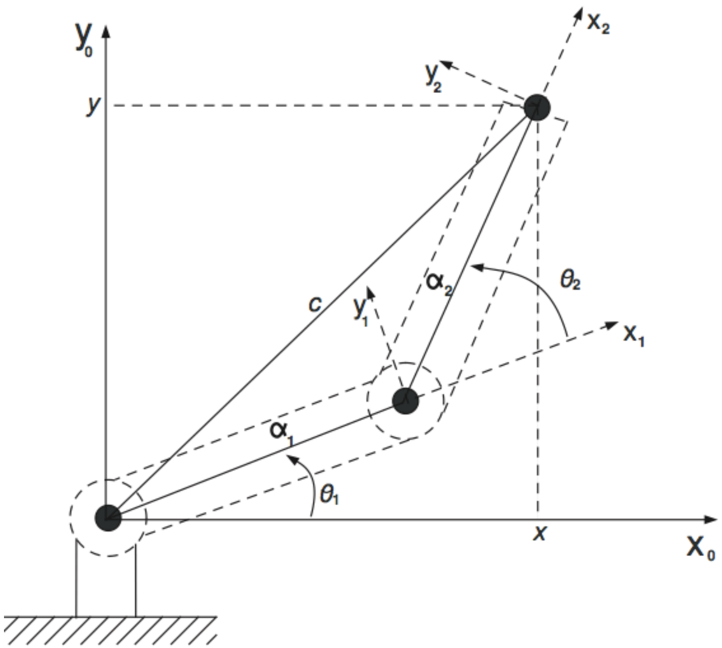
- Joint angles in terms of the Cartesian position of the end-effector

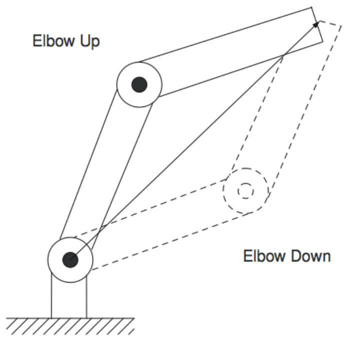
$$\cos\theta_2 = \frac{(x^2 + y^2 - \alpha_1^2 - \alpha_2^2)}{2\alpha_1\alpha_2} := D$$

$$\sin\theta_2 = \pm\sqrt{1 - D^2}$$

$$\theta_2 = \tan^{-1}\left(\frac{\pm\sqrt{1 - D^2}}{D}\right)$$

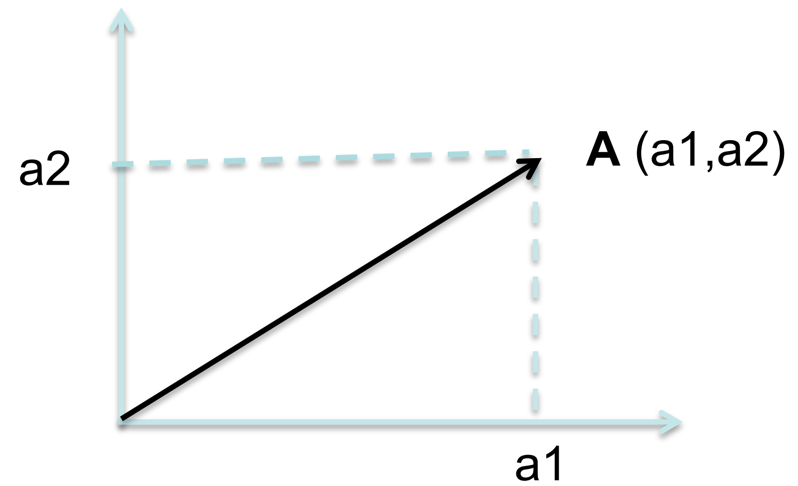
$$\theta_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{\alpha_2 \sin\theta_2}{\alpha_1 + \alpha_2\cos\theta_2}\right)$$



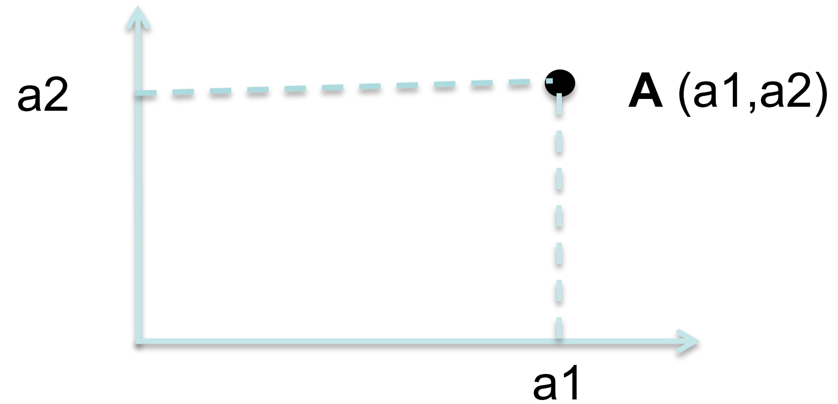


Data types

- 2D vector: $real^2$

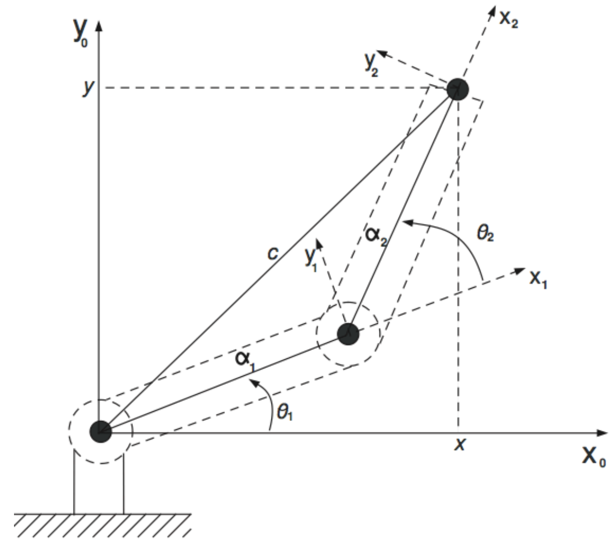


- A point in the Cartesian Plane: $real^2$



Formalization of the Manipulator

- Each link is modeled as a 2D-vector
- The manipulator is modeled as a sum of two vectors



Definition 1: Two Link Manipulator

$\vdash \forall$ A B. tl_manipulator A B = A + B

- *tl_manipulator:* $\mathtt{real}^2 \rightarrow \mathtt{real}^2 \rightarrow \mathtt{real}^2$

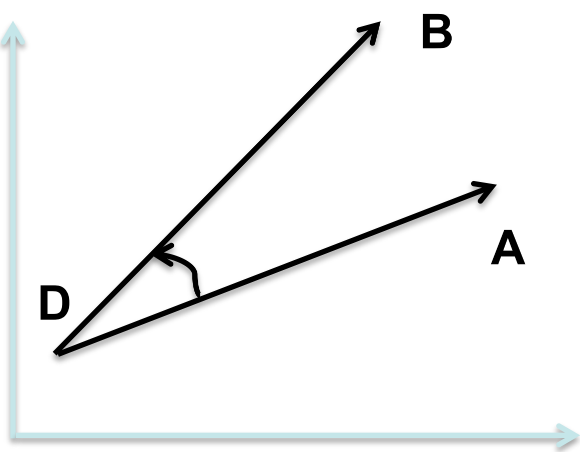
Direction of Rotation

- An angle is always less than 180 degrees (pi)
  - Can be clockwise or anticlockwise

Definition 2: Anticlockwise

$\vdash \forall$ B A D. anticlockwise D (A,B) <=> $0 \leq \mathrm{Im}\ (\frac{B-D}{A-D})$

- *anticlockwise:*
  $(\mathtt{real}^2 \rightarrow \mathtt{real}^2 \rightarrow \mathtt{real}^2 \rightarrow \mathtt{bool})$
- Im: imaginary part



Two Dimensional Angle

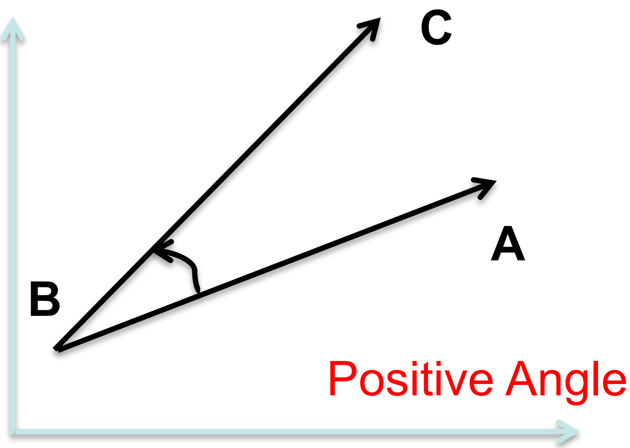
- More convenient to work with than the n-D angle

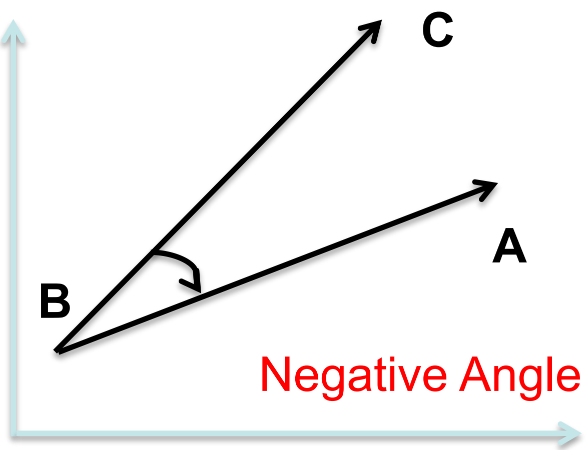
Definition 3: Plus Minus

$\vdash$ `plus_minus C = (if C then 1 else -1)`

Definition 4: Two Dimensional Angle

```
⊢ ∀ A B C. TDangle (A,B,C) =
        plus_minus (anticlockwise B (A,C)) * angle (A,B,C)
```

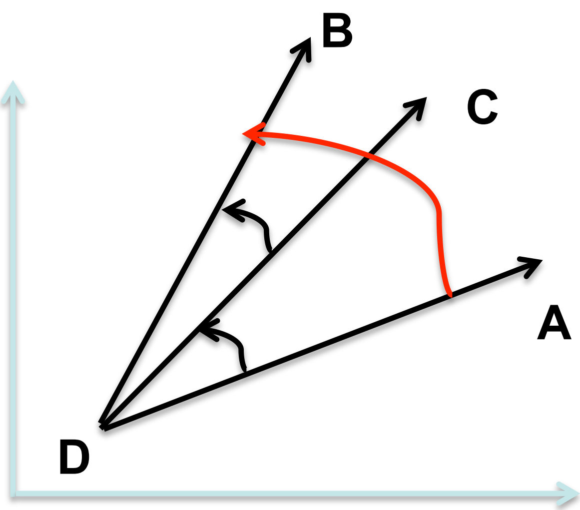




Angle Addition Theorem

Theorem 1: Angle Addition

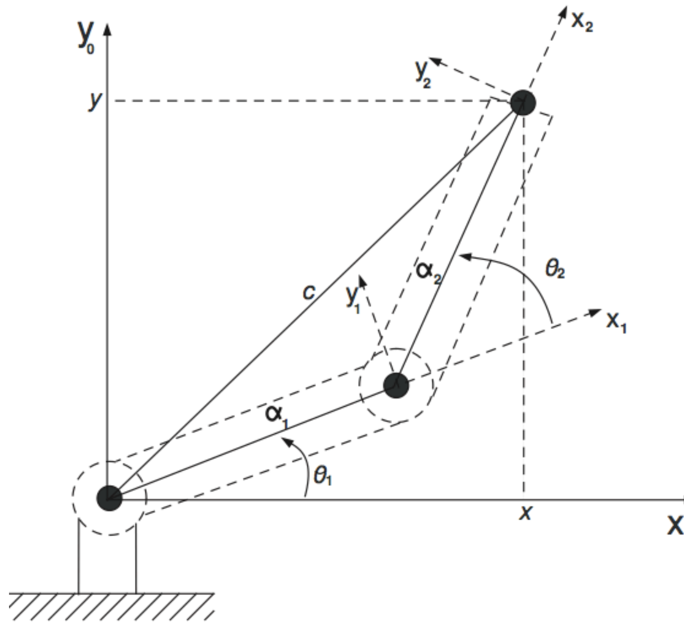
```
⊢ ∀ A B C D. ~(collinear {D, A, B}) ∧ ~(C = D) ∧
        anticlockwise D (A,C) ∧
        anticlockwise D (C,B) ∧
        anticlockwise D (A,B) ⇒
      angle  (A,D,B) = angle (A,D,C) + angle (C,D,B)
```



Formal Verification of Forward Kinematics

$$x = \alpha_1 \cos\theta_1 + \alpha_2 \cos(\theta_1 + \theta_2)$$

$$y = \alpha_1 \sin\theta_1 + \alpha_2 \sin(\theta_1 + \theta_2)$$



- Various cases related to the positioning of links in different quadrants have to be considered

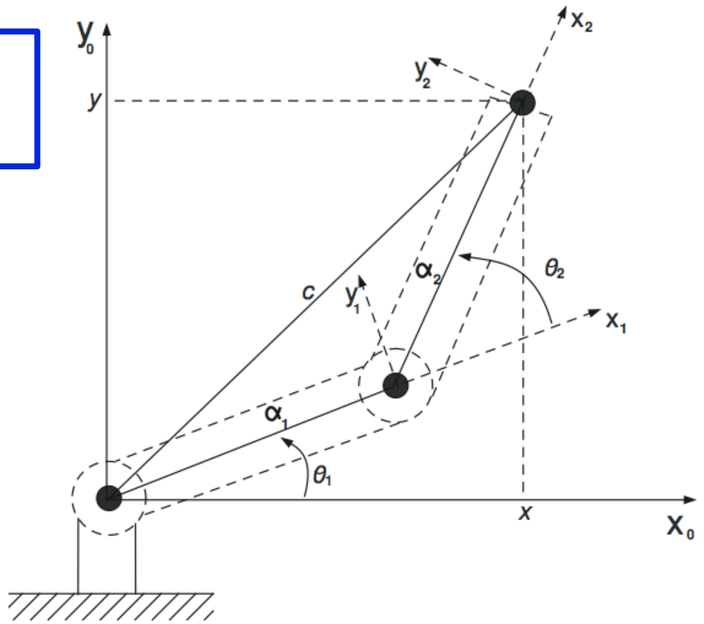
Theorem 2: X-Component with the First Link in the Upper Half Plane

```
⊢ ∀ A B. ~(A = vec 0) ∧ ~(B = vec 0) ∧
        anticlockwise (vec  0) (basis 1,A) ⇒
   (tl_manipulator A B)$1 =
     norm (A) * cos (vector_angle (basis 1) A) +
     norm (B) * cos (vector_angle A B + vector_angle (basis 1) A)
```

Formal Verification of Forward Kinematics

$$x = \alpha_1 \cos\theta_1 + \alpha_2 \cos(\theta_1 + \theta_2)$$

$$y = \alpha_1 \sin\theta_1 + \alpha_2 \sin(\theta_1 + \theta_2)$$



- Sub-Goal 2

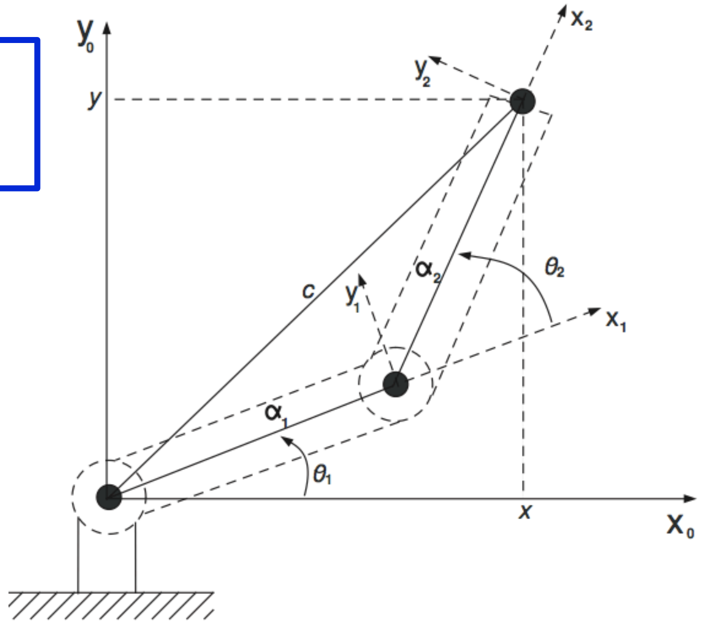
Theorem 3: X-Component with the First Link in the Lower Half Plane

```
⊢ ∀ A B. ~(A = vec 0) ∧ ~(B = vec 0) ∧
        ~(anticlockwise (vec 0) (basis 1,A)) ⇒
   (tl_manipulator A B)$1 =
     norm (A) * cos (vector_angle (basis 1) A) +
     norm (B) * cos (vector_angle A B - vector_angle (basis 1) A)
```

Formal Verification of Forward Kinematics

$$x = \alpha_1 \cos\theta_1 + \alpha_2 \cos(\theta_1 + \theta_2)$$

$$y = \alpha_1 \sin\theta_1 + \alpha_2 \sin(\theta_1 + \theta_2)$$



- Final Goal

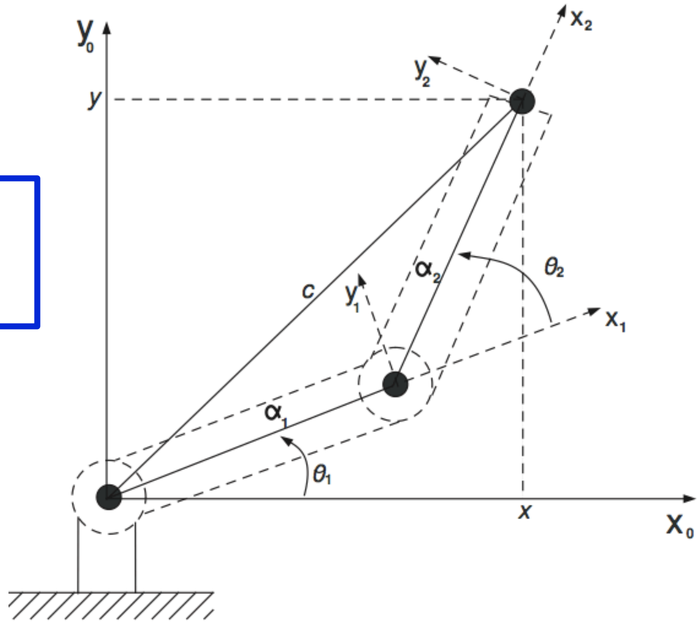
Theorem 4: X-Component

```
⊢ ∀ A B. ~(A = vec 0) ∧ ~(B = vec 0) ⇒
    (tl_manipulator A B)$1 =
      norm (A) * cos (TDangle (basis 1,vec 0,A)) +
      norm (B) * cos (TDangle (A,vec 0,B) +
                      TDangle (basis 1,vec 0,A))
```

Formal Verification of Forward Kinematics

$$x = \alpha_1 \cos\theta_1 + \alpha_2 \cos(\theta_1 + \theta_2)$$

$$y = \alpha_1 \sin\theta_1 + \alpha_2 \sin(\theta_1 + \theta_2)$$



- Final Goal

Theorem 5: Y-Component

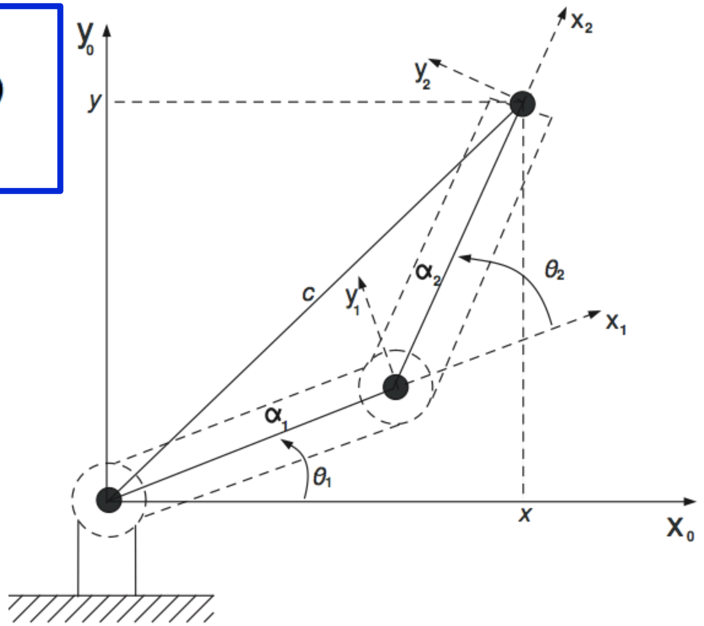
```
⊢ ∀ A B. ~(A = vec 0) ∧ ~(B = vec 0) ⇒
    (tl_manipulator A B)$2 =
      norm (A) * sin (TDangle (basis 1,vec 0,A)) +
      norm (B) * sin (TDangle (A,vec 0,B) +
                      TDangle (basis 1,vec 0,A))
```

Formal Verification of Inverse Kinematics

$$\cos\theta_2 = \frac{(x^2 + y^2 - \alpha_1^2 - \alpha_2^2)}{2\alpha_1\alpha_2} := D$$

$$\sin\theta_2 = \pm\sqrt{1 - D^2}$$

$$\theta_2 = \tan^{-1}\left(\frac{\pm\sqrt{1 - D^2}}{D}\right)$$



Theorem 6: cos θ_2

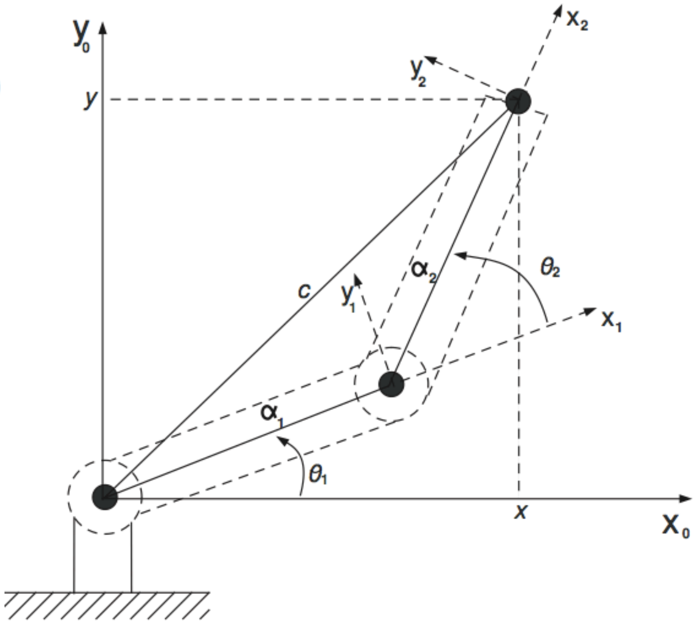
```
⊢ ∀ A B. ~(A = vec 0) ∧ ~(B = vec 0) ∧ ~(A + B = vec 0) ⇒
      cos (vector_angle A B) =
         ((((tl_manipulator A B)$1) pow 2) +
          (((tl_manipulator A B)$2) pow 2) - (norm (A) pow 2) -
              (norm (B) pow 2)) / (2 * norm (A) * norm (B))
```

Formal Verification of Inverse Kinematics

$$\cos\theta_2 = \frac{(x^2 + y^2 - \alpha_1^2 - \alpha_2^2)}{2\alpha_1\alpha_2} := D$$

$$\sin\theta_2 = \pm\sqrt{1 - D^2}$$

$$\theta_2 = \tan^{-1}\left(\frac{\pm\sqrt{1 - D^2}}{D}\right)$$



Theorem 7: sin θ_2

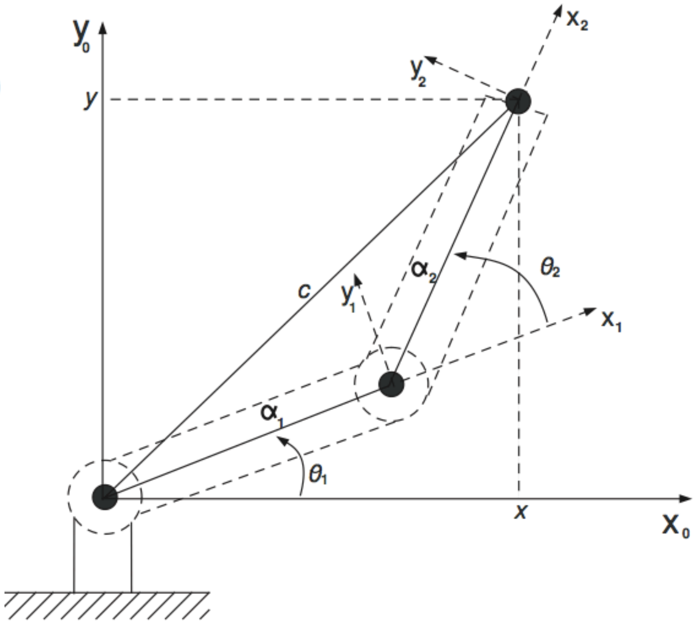
```
⊢ ∀ A  B. sin (vector_angle  A  B) =
          sqrt(1 - cos (vector_angle  A  B) pow 2)
```

Formal Verification of Inverse Kinematics

$$\cos\theta_2 = \frac{(x^2 + y^2 - \alpha_1^2 - \alpha_2^2)}{2\alpha_1\alpha_2} := D$$

$$\sin\theta_2 = \pm\sqrt{1 - D^2}$$

$$\theta_2 = \tan^{-1}\left(\frac{\pm\sqrt{1 - D^2}}{D}\right)$$



Theorem 8: θ_2

```
⊢ ∀ A B. ~(A = vec 0) ∧ ~(B = vec 0) ∧ ~(A + B = vec 0) ⇒
      vector_angle A B =
       atn ((sqrt (1 - (cos (vector_angle A B) pow 2))) /
          cos (vector_angle A B))
```

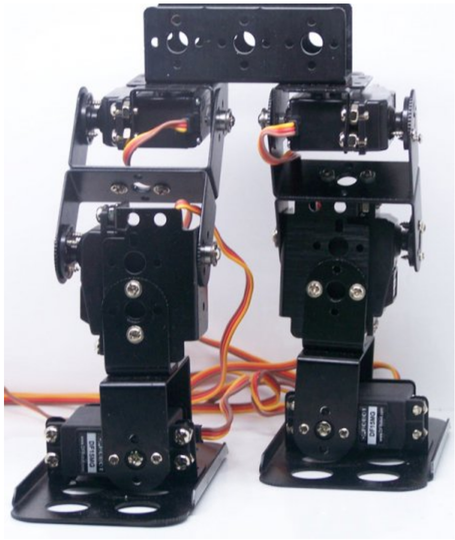
Proof Experience

- Extensive human guidance required
  - 15,000 lines of code (700 man-hours)

- Generic nature of the HOL-Light's Geometry theory was very useful

Case Study: Biped Robot

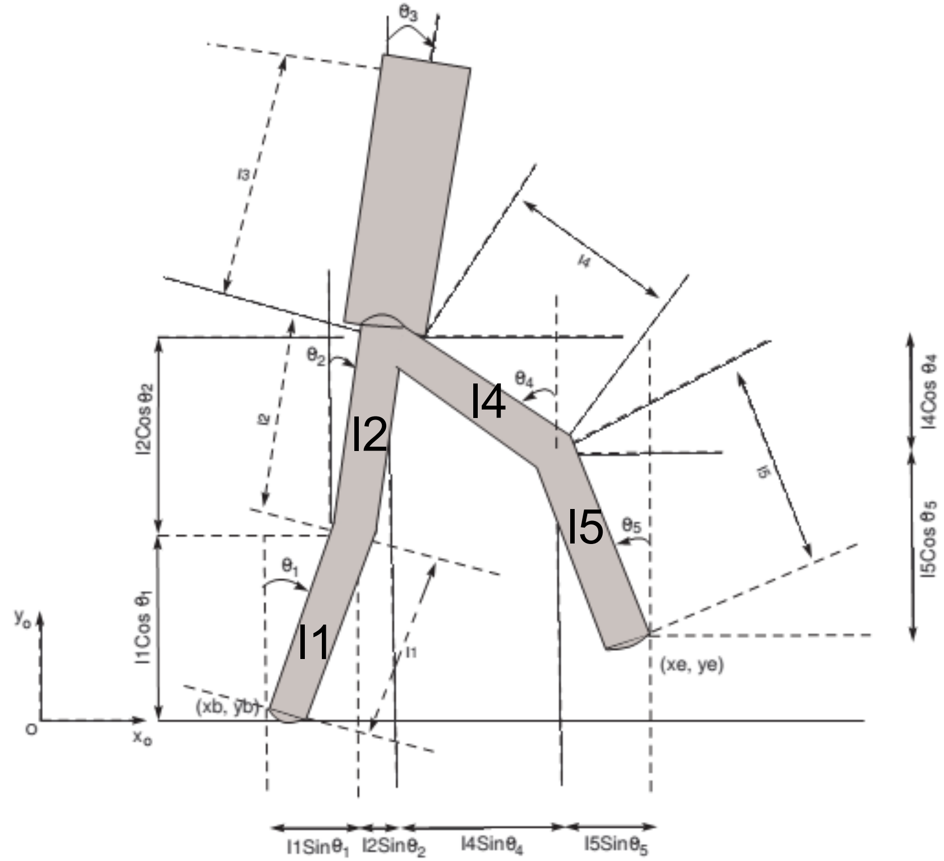
- Two-legged walking robot (Human-like mobility)
  - More efficient than the conventional wheeled robots for maneuvering fields with ladders, stairs, and uneven surfaces



- Classical case study in the robotics community

Formal Model

- 3 Joints: hip, two knees
- 4 Links
  - upper legs: l2 and l4
  - lower legs: l1 and l5



Definition 5: Biped Robot

```
⊢ ∀ A B C D. biped A B C D =
        tl_manipulator (tl_manipulator A B) (tl_manipulator C D)
```

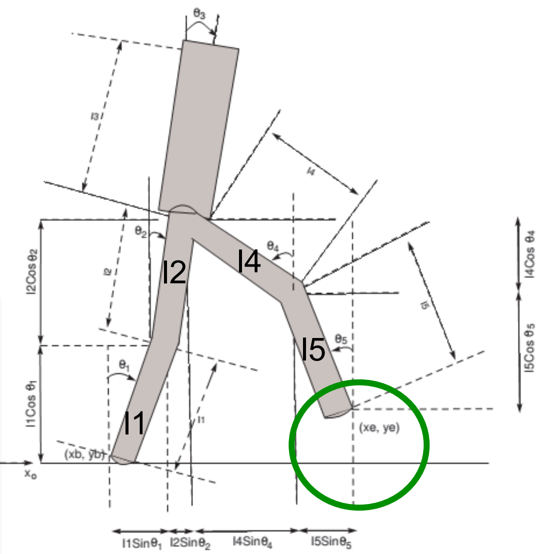
Formal Kinematic Analysis – Biped Robot

$$x_e = l_1 \sin\theta_1 + l_2 \sin\theta_2 + l_4 \sin\theta_4 + l_5 \sin\theta_5$$

$$y_e = l_1 \cos\theta_1 + l_2 \cos\theta_2 - l_4 \cos\theta_4 - l_5 \cos\theta_5$$

Theorem 8: x-Component of Biped Robot

```
⊢ ∀ l1 l2 l4 l5.
~(l1 = vec 0) ∧ ~(l2 = vec 0) ∧ ~(l4 = vec 0) ∧ ~(l5 = vec 0) ∧
  anticlockwise (vec 0) (l1,l2) ∧ anticlockwise (vec 0) (l5,l4) ∧
        anticlockwise (vec 0) (basis 1,l1) ∧
        anticlockwise (vec 0) (l1,basis 2) ∧
        anticlockwise (vec 0) (basis 1,l2) ∧
        anticlockwise (vec 0) (l2,basis 2) ∧
        anticlockwise (vec 0) (basis 1,l5) ∧
        anticlockwise (vec 0) (basis 2,l5) ∧
        anticlockwise (vec 0) (basis 2,l4) ∧
        ~ collinear {basis 1, vec 0, l5} ∧
        ~ collinear {basis 1, vec 0, l2} ∧
        ~ collinear {basis 2, vec 0, l4} ⇒
           (biped l1 l2 l4 l5)$1 =
             norm (l1) * sin (vector_angle (basis 2) l1) +
             norm (l2) * sin (vector_angle (basis 2) l2) -
             norm (l4) * sin (vector_angle (basis 2) l4) -
             norm (l5) * sin (vector_angle (basis 2) l5)
```

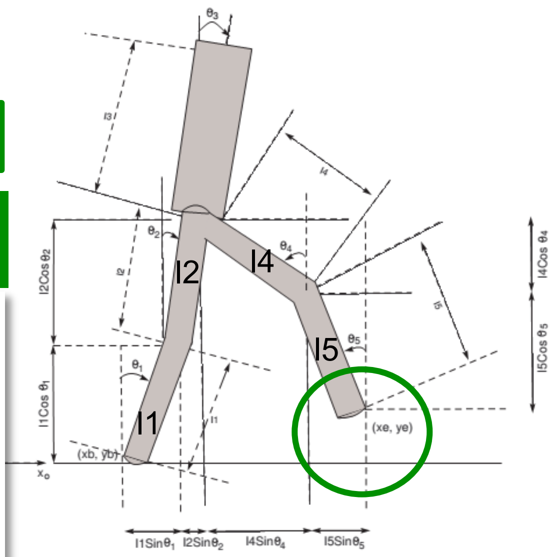


H. K. Lum et al. Planning and Control of a Biped Robot. International J. of Engineering Science, 37:1319–1349, 1999.

Formal Kinematic Analysis - Biped Robot

$$x_e = l_1 \sin\theta_1 + l_2 \sin\theta_2 + l_4 \sin\theta_4 + l_5 \sin\theta_5$$

$$y_e = l_1 \cos\theta_1 + l_2 \cos\theta_2 - l_4 \cos\theta_4 - l_5 \cos\theta_5$$



H. K. Lum et al. Planning and Control of a Biped Robot. International J. of Engineering Science, 37:1319–1349, 1999.

Theorem 9: y-Component of Biped Robot

```
⊢ ∀ l1 l2 l4 l5.
~(l1 = vec 0) ∧ ~(l2 = vec 0) ∧ ~(l4 = vec 0) ∧ ~(l5 = vec 0) ∧
  anticlockwise (vec 0) (l1,l2) ∧ anticlockwise (vec 0) (l5,l4) ∧
       anticlockwise (vec 0) (basis 1,l1) ∧
       anticlockwise (vec 0) (l1,basis 2) ∧
       anticlockwise (vec 0) (basis 1,l2) ∧
       anticlockwise (vec 0) (l2,basis 2) ∧
       anticlockwise (vec 0) (basis 1,l5) ∧
       anticlockwise (vec 0) (basis 2,l5) ∧
       anticlockwise (vec 0) (basis 2,l4) ∧
       ~ collinear {basis 1, vec 0, l5} ∧
       ~ collinear {basis 1, vec 0, l2} ∧
       ~ collinear {basis 2, vec 0, l4} ⇒
          (biped  l1  l2  l4  l5)$2 =
            norm (l1) * cos (vector_angle (basis 2) l1) +
            norm (l2) * cos (vector_angle (basis 2) l2) +
            norm (l4) * cos (vector_angle (basis 2) l4) +
            norm (l5) * cos (vector_angle (basis 2) l5)
```

Case Study: Biped Robot

- The mismatches in the results highlight the dire need of a sound kinematic analysis technique

- The proofs were very straightforward
  - 200 Lines of code (A week time: mostly spent on double checking the analysis)
    - Indicates the usefulness of our foundational work

- All assumptions are always guaranteed to be accompanying the formally verified Theorems
  - Not the case in other approaches

Conclusions

- A generic formal framework to reason about the Kinematic Analysis of Two-link Planar Manipulator
  - No compromise on the accuracy of the model or analysis (useful for safety-critical applications)

- Case Study on the kinematic analysis of the Biped Robot demonstrates the practical utilization of the proposed framework.

Future Directions

- Can be built upon to conduct formal kinematic analysis of other robotic manipulators
  - Kinematic Analysis of Selective Compliant Assembly Robot Arm (SCARA)
- Develop more advanced Kinematic Analysis foundations
  - Extend the coordinate frame from 2D to 3D and formally verify the Denavit Hartenberg (DH) parameters
    - Further Enhance the scope of Formal Kinematic Analysis

Thanks!

- For More Information
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    - http://save.seecs.nust.edu.pk
  - Contact
    - osman.hasan@seecs.nust.edu.pk