Formalization of Laplace Transform using the Multivariable Calculus Theory of HOL-Light

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Outline

- Introduction
- Motivation
- Formalization Details
- Case Study
  - Linear Transfer Converter (LTC) circuit
- Conclusions

Laplace Transform

- Integral transform method
  - *Pierre Simon Laplace 1749-1827*

- Mathematically represented by the following improper integral

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt, \;\; s \in \mathbb{C}$$

- A linear operator
  - Input: Time varying function, i.e., a function $f(t)$ with a real argument t ($t \geq 0$)
  - Output: $F(s)$ with complex argument s

Laplace Transform - Key Benefits and Utilizations

- Solve linear Ordinary Differential Equations (ODEs) using simple algebraic techniques

- Obtain concise and useful input/output relationships (Transfer Functions) for systems
  - Widely used in Control System and Analog Circuit Design

Laplace Transform - Example

$$\frac{d^2y(t)}{dt^2} + 4y(t) = x(t)$$

$$\mathcal{L}\Big(\frac{d^2y}{dt^2} + 4y(t)\Big)s = \mathcal{L}x(t)s$$

Taking Laplace Transform on Both sides

$$s^2(\mathcal{L}y(t)s) + 4(\mathcal{L}y(t)s) = (\mathcal{L}x(t)s)$$

Using the Laplace of a differential and the Linearity of Laplace Properties

$$\frac{(\mathcal{L}y(t)s)}{(\mathcal{L}x(t)s)} = \frac{1}{s^2+4}$$

Transfer Function

$$\text{let } x(t) = sin(2t), \text{ then } \mathcal{L}x(t)(s) = \frac{2}{s^2+4}$$

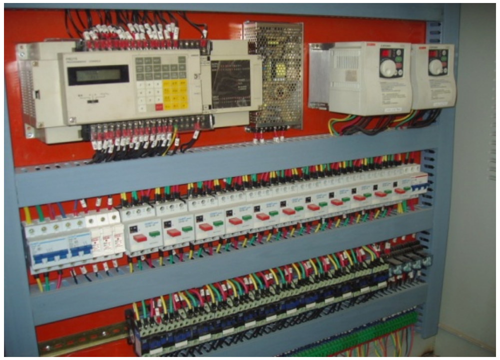
Laplace of sine

$$\mathcal{L}y(t)s = \frac{2}{(s^2+4)^2} \xrightarrow{\text{Inverse Laplace}} y(t) = \frac{1}{8}sin(2t) - \frac{t}{4}cos(2t)$$

Solution in time domain

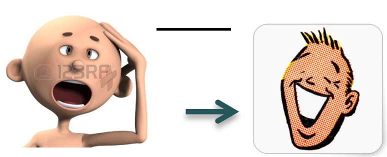
Real-World Applications for Laplace Transforms

- Integral part of analyzing many physical systems
  - Aerodynamic systems
  - Circuit Analysis
  - Control systems
  - Mechanical networks
  - Analogue filters

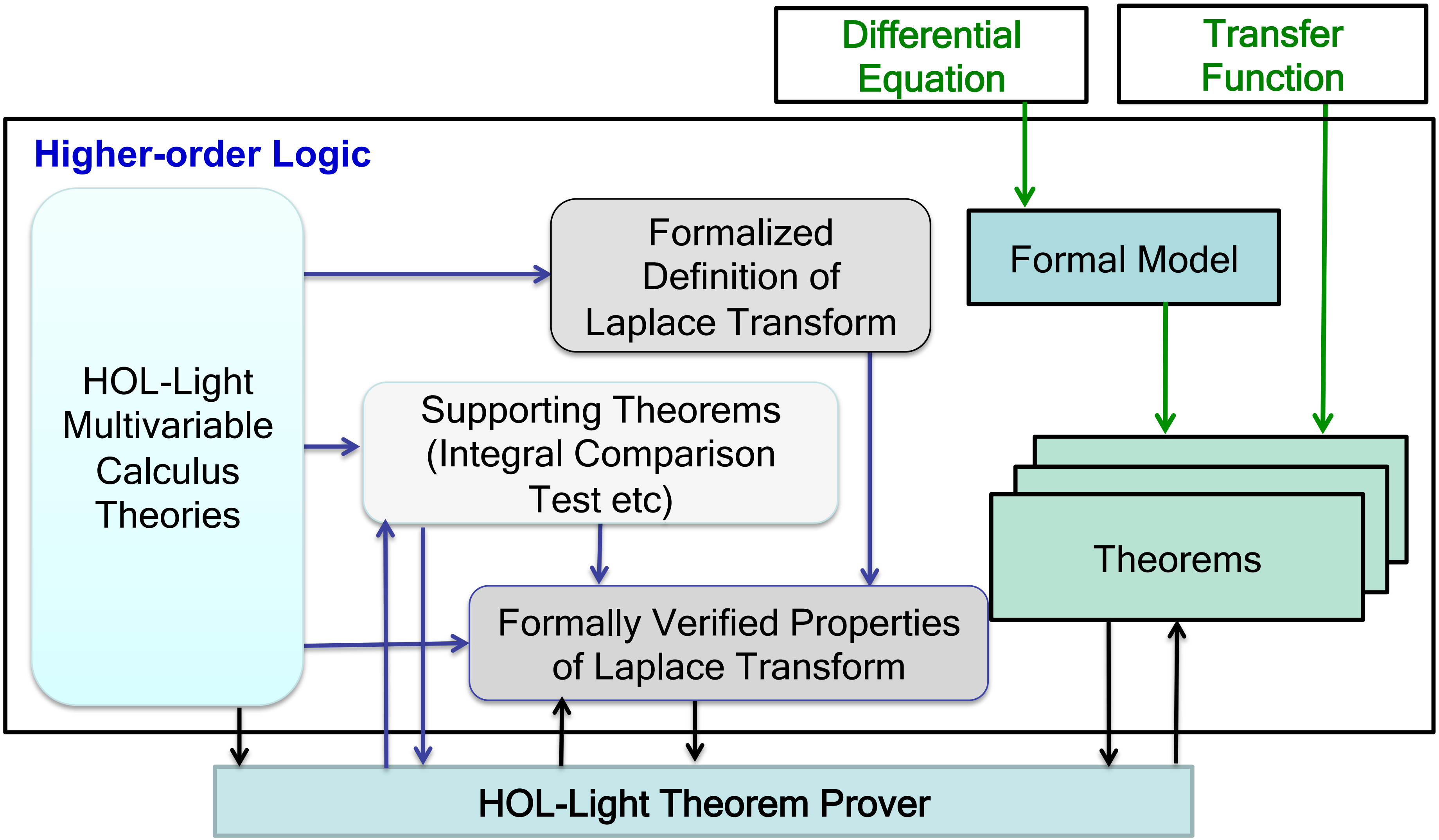


Laplace Transform based Analysis

Criteria	Paper-and-Pencil Proof	Simulation/ Symbolic Methods	Automated Formal Methods (MC, ATP)	Computer Algebra Systems	Higher-order-logic Proof Assistants
Expressiveness	✓	✓	✗	✓	✓
Accuracy	✓ ?	✗	✓	✗	✓
Automation	✗	✓	✓	✓	✗



Proposed Approach for Verifying Transfer Functions



Formal Definition of Laplace Transform

❑ Mathematical definition

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt,\ \ s \in \mathbb{C} \Longrightarrow F(s) = \lim_{b \to \infty} \int_0^b f(t)e^{-st}dt$$

Given f is piecewise smooth and is of exponential order i.e. there exist constants $\alpha \in \mathbb{R}$ and $0 < M$ such that $|f(t)| \leq Me^{\alpha t}$ for all $t \geq 0$

Definition : Laplace Transform

```
⊢ ∀ s f. laplace f s =
  lim at_posinfinity (λb. integral (interval [lift(&0),lift(b)])
    (λt. cexp (-(s * Cx(drop t))) * f t))
```

Definition : Conditions for Laplace Existence

```
⊢ ∀ s f. laplace_exists f s ⇔
(∀ b. f piecewise_differentiable_on interval [lift (&0),lift b])
  ∧ (∃ M a. Re s > drop a ∧ exp_order f M a)
```

Formalized Laplace Transform Properties

Property	Mathematical Form
Limit Existence	$\exists l. \int_0^\infty f(t)e^{-st}dt = l$
Linearity	$\mathcal{L}\left(\alpha f(t) + \beta g(t)\right)s = \alpha(\mathcal{L}f(t)s) + \beta(\mathcal{L}g(t)s)$
Frequency Shifting	$\mathcal{L}\left(e^{bt}f(t)\right)s = \mathcal{L}\ f(t)(s-b)$

Property	Formalized Form
Limit Existence	⊢ ∀ f s. laplace_exists f s ⇒ (∃l. ((λb. integral (interval [lift (&0),lift b]) (λt. cexp (-(s * Cx (drop t))) * f t)) → l) at_posinfinity)
Linearity	⊢ ∀ f g s a b. laplace_exists f s ∧ laplace_exists g s ⇒ laplace (λx. a * f x + b * g x) s = a * laplace f s + b * laplace g s
Frequency Shifting	⊢ ∀ f s b. laplace_exists f s ⇒ laplace (λt. cexp (b * Cx (drop t)) * f t) s = laplace f (s - b)

Formalized Laplace Transform Properties

Property	Mathematical Form
Integration	$\mathcal{L}(\int_0^t f(\tau)d\tau)s = \frac{1}{s}(\mathcal{L}f(t)s)$
n-order Differentiation	$\mathcal{L}(\frac{d^n f(t)}{dx^n})s = s^n(\mathcal{L}f(t)s) - \sum_{k=1}^{n} s^{k-1}\frac{d^{n-k}f(0)}{dx^{n-k}}$

Property	Formalized Form
Integration	⊢ ∀ f s. (&0 < Re s) ∧ laplace_exists f s ∧ laplace_exists (λx. integral (interval [lift (&0),x]) f) s ∧ (∀x. f continuous_on interval [lift (&0),x]) ⇒ laplace (λx. integral (interval [lift (&0),x]) f) s = inv(s) * laplace f s
Higher Order Differentiation	⊢ ∀ f s n. laplace_exists_higher_derivative n f s ∧ (∀x. higher_derivative_differentiable n f x) ⇒ laplace (λx. higher_order_derivative n f x) s = s pow n * laplace f s - vsum (1..n) (λx. s pow (x-1) * higher_order_derivative (n-x) f (lift (&0)))

- 5000 lines of HOL-Light code and approximately 800 man-hours

Case Study: Linear Transfer Converter (LTC) circuit

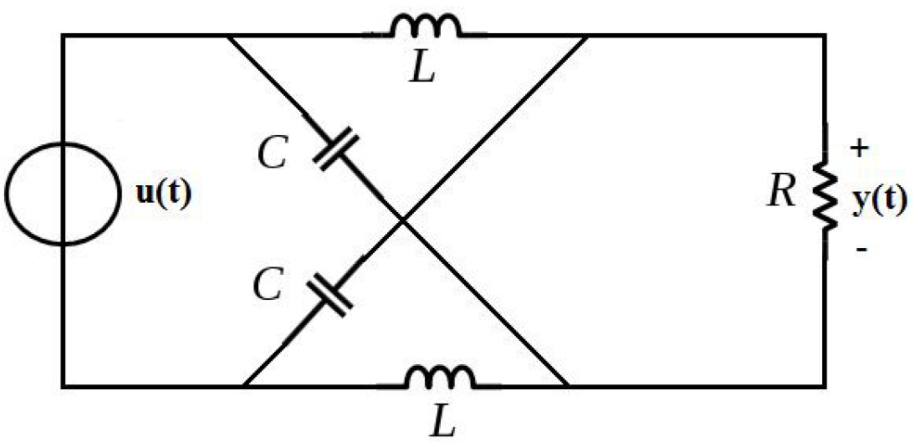
- Converts the voltage and current levels in power electronics systems
- Functional correctness of power systems depends on design and stability of LTC

Differential Equation:

$$\frac{d^2y}{dt^2} - \frac{2}{RC}\frac{dy}{dt} + \frac{1}{LC}y = \frac{d^2u}{dt^2} - \frac{1}{LC}u$$

Transfer Function:

$$\frac{Y(s)}{U(s)} = \frac{s^2 - \frac{1}{LC}}{s^2 - \frac{2s}{RC} + \frac{1}{LC}}$$



Linear Transfer Converter (LTC) circuit

Differential Equation: $$\frac{d^2y}{dt^2} - \frac{2}{RC}\frac{dy}{dt} + \frac{1}{LC}y = \frac{d^2u}{dt^2} - \frac{1}{LC}u$$

Definition : Differential Equation of LTC

```
⊢ ∀ y u x L C R. diff_eq_LTC y u x L C R ⇔
diff_eq 2 [ Cx (&1 / L * C); --Cx (&2 / R * C); Cx (&1)] y x =
diff_eq 2 [ --Cx (&1 / L * C ); Cx (&0); Cx (&1)] u x
```

Definition : Differential Equation

```
⊢ ∀ n L f x. diff_eq n L f x ⇔
  vsum (0..n) (λt. EL t L x * higher_order_derivative t f x)
```

Linear Transfer Converter (LTC)

Theorem : Transfer Function of LTC

```
⊢ ∀ y u s R L C. (&0 < R) ∧ (&0 < L) ∧ (&0 < C) ∧
  (zero_initial_conditions 1 u) ∧ ( zero_initial_conditions 1 y) ∧
   (∀x. higher_derivative_differentiable 2 y x) ∧
   (∀x. higher_derivative_differentiable 2 u x) ∧
   (higher_derivative_laplace_exists 2 y s) ∧
   (higher_derivative_laplace_exists 2 u s) ∧
  ~((Cx(&1/(L*C)) - Cx(&2/(R*C))*s) + s pow 2 = Cx(&0) )∧
  ~(laplace u s = Cx(&0)) ∧ (∀t. diff_eq_LTC y u t L C R) ⇒
   (laplace y s / laplace u s =
    (s pow 2 - Cx(&1/(L*C))) / ((Cx(&1/(L*C)) -
      Cx(&2/(R*C))*s) + s pow 2))
```

- 650 lines of HOL-Light code and the proof process took just a couple of hours

Conclusions

- Formalization of Laplace transform theory using higher-order logic
  - Multivariable Calculus Theory of HOL-Light
- Advantages
  - Accurate Results
  - Reduction in user-effort while formally analyzing Physical Systems that involve Differential Equations
- Case Study: Transfer function verification of LTC circuit

Future Directions

- Application of Laplace transform theory in Analog and Mixed Signal circuits and controls engineering

- Formalization of Inverse Laplace transform

- Formalization of Fourier transform

Thanks!

- For More Information
  - Visit our website
    - http://save.seecs.nust.edu.pk
  - Contact
    - osman.hasan@seecs.nust.edu.pk

Additional slides

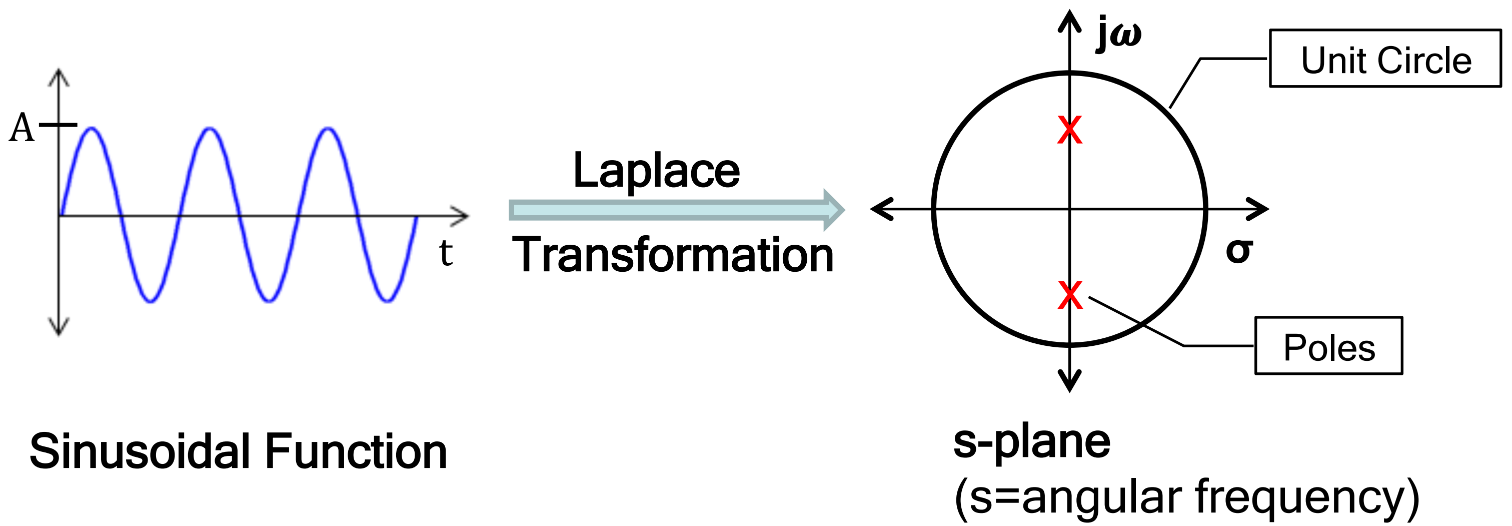
Formalized Laplace Transform

Definition 3: Exponential Order Function

$\vdash \forall$ f M a. exp_order f M a $\Leftrightarrow$ &0 $<$ M $\wedge$
($\forall$ t. &0 $\leq$ t $\Rightarrow$ norm (f (lift t)) $\leq$ M * exp (drop a * t))

Laplace Transform

- Provides compact representation of the overall behavior of the given time varying function



- s-plane representation depicts frequency and phase of sinusoidal signal

HOL-Light

- Multivariable calculus theories
  - Integral theory
  - Differential theory
  - Transcendental theory
  - Topological theory
  - Complex analysis theory
- Real number theory
- Natural number theory

Limit Existence of Laplace Transform

❑ Proof Steps

Split the complex integrand into real and imaginary parts

Convert both complex integrals to their corresponding real integral and split the complex limit to both integrals

Using fo… compari… converg…

In our…

```
∃l.( (λb. integral (interval [lift (&0),lift b])
     (λt. Cx (Re (cexp (-(s * Cx (drop t))) * f t))) +
  ii * integral (interval [lift (&0),lift b])
     (λt. Cx (Im (cexp (-(s * Cx (drop t))) * f t)))) → l)
      at_posinfinity
```

```
laplace_exists f s ⇒
  ∃k. ((λb. real_integral (real_interval [&0,b])
     (λx. abs (Re (cexp (-s * Cx (x)) * f(lift x))))) → k)
       at_posinfinity
```

Lemma 3: Comparison Test for Improper Integrals

```
∧ f x ≤ g x) ∧
) ∧
val [&0,t]) ⇒) ∧
g)⇒ k)
,b]) f) ⇒ k)
```

Lemma 2: Limit of a Complex-Valued Function … mplex Integral

```
⊢ ∀ f L1 L2.
  ((λt. Re (f t)) ⇒ L1) at_posinfinity ∧
  ((λt. Im (f t)) ⇒ L2) at_posinfinity ⇒
    (f → complex (L1,L2)) at_posinfinity
```