Formal Stability Analysis of Two-dimensional Digital Image Processing Filters

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Abstract. There are several image-processing applications that require to partition frequency components of images. This requirement is usually fulfilled by using digital image processing filters. Most of this processing is done in two dimensions given the two-dimensions of the regular images. It is very important to ascertain that these filters provide a stable output for a bounded input and this requirement is usually termed as stability. The stability analysis of these filters is usually conducted analytically, on a piece of paper, or by simulations. However, these techniques provide approximate or inaccurate results as paper-based analysis can have human error and simulations suffer from computer arithmetic related roundoff limitations. We advocate formally analysing the stability of digital filters for two-dimensional (2D) images using interactive theorem proving. In this regard, we present a formal dynamical model and a formal notion of stability of 2D digital image processing filters in HOL Light. The proposed formal model is used to perform the stability analysis of a real-world 2^{nd}-order filter in HOL Light.

Keywords: Stability, 2D z-transform, Interactive Theorem Prover

1 Introduction

Digital image-processing filters (IPFs) are extensively being used in many application areas, such as medicine [1] and autonomous vehicles [2, 9], for performing different operations, like image processing, filtering and enhancement, in Two-dimensional (2D) images. For example, they are used to pre and post process images to filter elements like noise and image smoothing and quality enhancement by filtering out the noise and distortion [1]. Filters are mainly of three kinds, namely, lows, band and high pass. For example, a high-pass filter can be used for the passage of frequencies higher than a certain range.

Stability of a digital IPF asserts a stable output response to a given bounded input and is considered as an important phenomenon for accessing the performance of an IPF. For a 2D digital IPF, it is described in terms of the transfer function, i.e., the relationship of output to input in the frequency domain. To analyze the stability analysis of a 2D digital IPFs, we first need to capture their dynamical behaviour in terms of 2D difference-equations (DEs). Next, the 2D z-transform is utilized for their analytical analysis by converting the DEs to algebraic equations, i.e., transforming the 2D arrays to the (z_1,z_2)-domain. Lastly, these (z_1,z_2)-domain representations are utilized for the stability analysis [13].

Conventionally, the stability analysis of the digital IPFs have been conducted using analytically on paper or simulations. But these methods, due to their human-error proneness and round off errors, cannot guarantee accurate results. Therefore, these conventional approaches cannot be relied upon considering the wider utility of these filters in many critical domains, like transportation and healthcare.

Formal verification [8] is an analysis approach that involves capturing the behavior of the given system in the form of a logical model and verifying the system characteristics deductively in a computer. Interactive theorem proving [4, 7] is one of the extensively used formal verification techniques. We argue to use interactive theorem proving to conduct the stability analysis of the digital IPFs. In this regard, we formalize the dynamics of the digital IPF as a 2D array in the HOL Light prover [6] . This model is then used to perform the formal stability analysis based on the transfer function, obtained using the z-transform on the dynamical model of the digital filter. We chose HOL Light for our work as it has a strong reasoning support for multivariate calculus and digital IPFs [12]. These existing works have greatly facilitated our formalization as we built upon them to develop reasoning support for the stability analysis of digital IPF.

We introduce HOL Light and some of definitions and some of the utilized theorems from HOL Light's theory of the multivariable calculus in Section 2. Section 3 describes the modeling of 2D z-transform in HOL Light. The formal model for stability of the 2D digital IPFs in presented in Section 4. The formal stability analysis of the 2^{nd}-order digital IPF is described in Section 5. Finally, Section 6 provides some insights that we gathered from our work as well our plans to further extend out reasoning support.

2 Preliminaries

We present some background information in this section to help the reader in understanding the remaining paper.

2.1 Interactive Theorem Prover: HOL Light

HOL Light [5], developed using ML [11], is an interactive proof-assistant that is widely used for developing proofs for the mathematical concepts and analyzing software and hardware systems. A theorem is a mathematical statement that can be proved using a predefined set of primitive rules or axioms in a theorem prover, ensuring the soundness of the proof development environment. HOL Light contains several multivariate theories, in particular, vectors, differential, integral and 2D z-transform, which are used in the proposed work.

2.2 Multivariable Calculus

A generic vector is modeled as a N element matrix, i.e., $\mathbb{R}^N$, of real numbers. This model allows us to use matrix operations for vector manipulations.

Summation over a generalized function f of an arbitrary data-type $\mathsf{A} \rightarrow \mathbb{R}^N$ is modeled as:

Definition 1. $\vdash_{def}$ ∀s f. vc_smm s f = (lambda j. smm s (λy. f y$j))

where vc_smm accepts an arbitrary set s: and a function f as inputs and outputs the vector-addition on s, which represents a set. smm models a finite summation over f and thus vc_smm (0..n) f mathematically models $\sum_{j=0}^{n} f(j)$. Similarly, we formalize the mathematical expression $\sum_{j=0}^{\infty} f(j) = l$, involving an infinite summation for a function f of datatype $\mathbb{N} \rightarrow \mathbb{R}^N$ and a limit value l of data-type $\mathbb{R}^N$, in HOL Light as follows:

Definition 2. $\vdash_{def}$ ∀s f l. (f smms l) s ⇔
((λn. vc_smm (s ∩ (0..n)) f) → l) squntialy

where, squntialy mathematically represents a sequential growth, i.e., $f(j), f(j+1), ...,$ etc.

Definition 3. $\vdash_{def}$ ∀f s. smmble f s ⇔ (∃l. (f smms l) s)

The HOL Light function smmble mathematically models $\sum_{j=0}^{\infty} f(j) = l$.

Next, we present the formal modeling of the infinite summation:

Definition 4. $\vdash_{def}$ ∀f s. inft_smm s f = (∈l. (f smms l) s)

where the return value l: $\mathbb{R}^N$ is the value of the infinite summation of the converging function f from the given starting point s.

3 Formal Modeling of the 2D z-Transform

The z-transform of a 2D discrete-time function $f(m_1, m_2)$ is expressed as [13]:

$$F(z_1, z_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} f(m_1, m_2) {z_1}^{-m_1} {z_2}^{-m_2} \tag{1}$$

We formalize Equation (1) as follows:

Definition 5. $\vdash_{def} \forall \mathsf{f}\ \mathsf{z_1}\ \mathsf{z_2}.\ \mathsf{z\_2d\_trnsfm}\ \mathsf{f}\ \mathsf{z_1}\ \mathsf{z_2} = \mathsf{inft\_smm}\ (\mathsf{from}\ 0)$
$$\left(\lambda \mathsf{m_1}.\ \mathsf{inft\_smm}\ \ (\mathsf{from}\ \ 0)\ \left(\lambda \mathsf{m_2}.\ \frac{\mathsf{f}\ \mathsf{m_1}\ \mathsf{m_2}}{\mathsf{z_1}^{\mathsf{m_1}}\ *\ \mathsf{z_2}^{\mathsf{m_2}}}\right)\right)$$

Here, we need to identify the set of all values of z_1 and z_2 for which the infinite summations converge to some finite value and thus ensure a finite $F(z_1, z_2)$, commonly known as the Region of Convergence (ROC). We can mathematically express and formally model the ROC as follows:

$$\text{ROC} = z_1, z_2 \in \mathbb{C} : \exists k. \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} f(m_1, m_2) {z_1}^{-m_1} {z_2}^{-m_2} = k \qquad (2)$$

Definition 6. $\vdash_{def} \forall \mathsf{f}\ \mathsf{m_1}.\ \mathsf{z\_2d\_ROC}\ \mathsf{f}\ \mathsf{m_1} =$
$$\{(\mathsf{z_1},\ \mathsf{z_2}) \mid (\mathsf{z_1} \neq 0) \wedge (\mathsf{z_2} \neq 0) \wedge$$
$$\mathsf{z\_2d\_tr\_smmble}\ \mathsf{f}\ \mathsf{z_1}\ \mathsf{z_2}\ \mathsf{m_1} \wedge \mathsf{z\_2d\_tr\_td\_smmble}\ \mathsf{f}\ \mathsf{z_1}\ \mathsf{z_2}\}$$

where, z_2d_ROC takes a function f and $\mathsf{m_1}$, which represents the starting-point in Equation (1), as inputs and outputs a set of non-zero values of variables $\mathsf{z_1}$ and $\mathsf{z_2}$ for which the 2D z-transform of f exists. We also formalized functions z_2d_tr_smmble and z_2d_tr_td_smmble capturing the summability of the function f for the inner and the outer (double) summations, respectively as:

Definition 7. $\vdash_{def} \forall \mathsf{f}\ \mathsf{z_1}\ \mathsf{z_2}\ \mathsf{m_1}.\ \mathsf{z\_2d\_tr\_smmble}\ \mathsf{f}\ \mathsf{z_1}\ \mathsf{z_2}\ \mathsf{m_1} =$
$$\left(\forall \mathsf{m_1}.\ \mathsf{smmble}\ \ (\mathsf{from}\ \ 0)\ \left(\lambda \mathsf{m_2}.\ \frac{\mathsf{f}\ \mathsf{m_1}\ \mathsf{m_2}}{\mathsf{z_1}^{\mathsf{m_1}}\ *\ \mathsf{z_2}^{\mathsf{m_2}}}\right)\right)$$

Definition 8. $\vdash_{def} \forall \mathsf{f}\ \mathsf{z1}\ \mathsf{z2}.\ \mathsf{z\_tr\_td\_summable}\ \mathsf{f}\ \mathsf{z1}\ \mathsf{z2} = \mathsf{summable}\ (\mathsf{from}\ 0)$
$$\left(\lambda \mathsf{m_1}.\ \mathsf{inft\_smm}\ \ (\mathsf{from}\ \ 0)\ \left(\lambda \mathsf{m_2}.\ \frac{\mathsf{f}\ \mathsf{m_1}\ \mathsf{m_2}}{\mathsf{z_1}^{\mathsf{m_1}}\ *\ \mathsf{z_2}^{\mathsf{m_2}}}\right)\right)$$

Now, we formally verify some key characteristics of the 2D z-transform, including linearity, shifting, scaling, complex conjugation and 2D z-transform of a n-order system, in HOL Light. This formally verified characteristics play a key role in the proposed stability analysis of the 2D digital IPFs, as presented in Section 5. The 2D z-transform, presented in this section, has been formalized by Rashid et al. [12]. However, the authors have not performed the stability analysis of the 2D digital IPF, which is indeed the scope of this paper. The actual formalization of the 2D z-transform can be viewed at[3].

[3] `http://save.seecs.nust.edu.pk/fsadipf/`

4 Stability of a 2D Digital Image Processing System

Stability is considered as an important characteristic while designing a 2D digital IPF. A discrete-time system, such as a digital filter is said to be stable if it provides a bounded output for a given bounded input. An important condition for the stability of a linear shift invariant (LSI) system can be mathematically expressed as [10]:

$$\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} |h(m_1, m_2)| < \infty \tag{3}$$

where $h(n_1, n_2)$ provides the impulse response, i.e., the output response when the input is a brief input function, of the given LSI system. However, it is more convenient to represent stability based on the system function/transfer function $H(z_1, z_2)$ (the Laplace transform of $h(m_1, m_2)$), which is mathematically expressed as:

$$H(z_1, z_2) = \frac{Y(z_1, z_2)}{X(z_1, z_2)} \tag{4}$$

According to Shanks, the stability of a LSI system, such as, digital filter can be mathematically expressed by the two conditions as follows [10]:

$$\begin{aligned} \text{Stability} \Leftrightarrow \quad & (a)\ X(z_1, z_2) \neq 0 \ \text{ for } \ |z_1| = 1, |z_2| \geq 1 \\ \text{and} \quad & (b)\ X(z_1, z_2) \neq 0 \ \text{ for } \ |z_1| \geq 1, |z_2| = 1 \end{aligned} \tag{5}$$

We can use the following two steps to ensure Condition (a) of the stability for the digital IPF. In the first step, we need to solve for all (z_1, z_2), such that $X(|z_1| = 1, z_2) = 0$, which is equivalent to solving for all (ω_1, z_2), such that $X(e^{j\omega_1}, z_2) = 0$. In the next step, we have to check that if all $|z_2|$ obtained in the first step are less than 1. Similarly, we can use the similar steps to ensure Condition 2 of the stability. Using this alternative representation, we formalized the stability of a digital filter in HOL Light as follows:

Definition 9. $\vdash_{def} \forall$X. cnd1_stbl_dgtl_fltr X =
$\{(\omega_1,\ z_2) \mid X\ (e^{j\omega_1},\ z_2) = 0 \land |z_2| < 1\} \neq \{\ \}$
$\vdash_{def} \forall$X. cnd2_stbl_dgtl_fltr X = $\{(z_1,\ \omega_2) \mid X\ (z_1,\ e^{j\omega_2}) = 0 \land |z_1| < 1\} \neq \{\ \}$
$\vdash_{def} \forall$X. is_stbl_dgtl_fltr X = cnd1_stbl_dgtl_fltr X $\land$ cnd2_stbl_dgtl_fltr X

where is_stbl_dgtl_fltr accepts the denominator X of the transfer function, provided in Equation (4), corresponding to the dynamics of a digital IPF and provides a stable filter.

5 Formal Stability Analysis of a 2^{nd}-order Filter

We utilize the formalization, provided in Sections 3 and 4, for performing the formal stability analysis of a 2^{nd}-order 2D digital IPF in this section. This illustrates the practical utilization of the foundational formal modeling, presented in this paper.

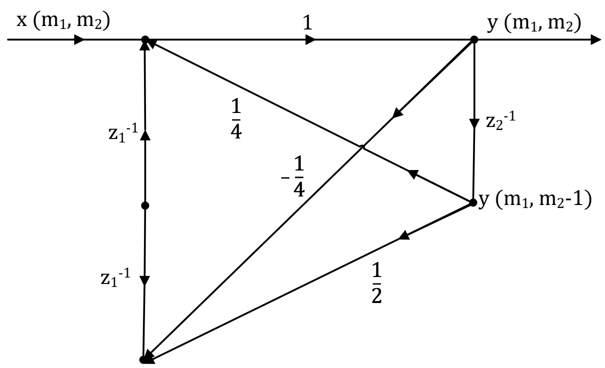

Fig. 1: Flowgraph of a 2^{nd}-order 2D IPF

Graphically, we can present a 2^{nd}-order 2D digital IPF by the flowgraph depicted in Figure 1. It is a collection of nodes and branches, which provide the directed connections between these nodes. The constants 1, $\frac{1}{4}$, $-\frac{1}{4}$ and $\frac{1}{2}$ in Figure 1 present the gains of each branches. Similarly, ${z_1}^{-1}$ and ${z_2}^{-1}$ model the horizontal and vertical delay, i.e., shift right and shift up, operations, respectively. This 2^{nd}-order digital IPF can be mathematically expressed using the following linear difference equation (DE).

$$y(m_1, m_2) = x(m_1, m_2) + \frac{1}{4}y(m_1, m_2 - 1) - \frac{1}{4}y(m_1 - 2, m_2) + \frac{1}{2}y(m_1 - 2, m_2 - 1) \tag{6}$$

We can mathematically describe the transfer function of the 2^{nd}order digital IPF corresponding to its dynamical model (Equation (6)) as follows:

$$H(z_1, z_2) = \frac{Y(z_1, z_2)}{X(z_1, z_2)} = \frac{1}{1 - \frac{1}{4}{z_2}^{-1} + \frac{1}{4}{z_1}^{-2} - \frac{1}{2}{z_1}^{-2}{z_2}^{-1}} \tag{7}$$

The main purpose of presentation this case study is to use our proposed formal models to formally verify Equation (7). To verify the transfer function, the first step is to formally model the DE of the filter (Equation (6)) as follows:

Definition 10. $\vdash_{def}$ ∀y x m1 m2 p q. dgtl_scnd_odr_fltr x y p q m1 m2 ⇔
y (m1, m2) = l1l2th_dfrnce_eq y p 2 2 m1 m2 - l1l2th_dfrnce_eq x q 0 0 m1 m2

with coefficients a and b of the input and output 2D arrays. The function dgtl_scnd_odr_fltr accepts the 2D arrays x and y, their coefficients a and b, and it uses the (L_1, L_2)-order DE l1l2th_dfrnce_eq to capture the linear DE expressing the 2^{nd}-order digital IPF.

We verify Equation (7) as follows:

Theorem 1. $\vdash_{thm}$ ∀x y p q z_1 z_2 m_1.
[C_1]: (z_1, z_2) IN 2d_roc_lccdifeq x y 2 2 q m_1 ∧
[C_2]: in_frst_qudrnt_2d_lccdifeq x y ∧
[C_3]: $z_1 \neq 0$ ∧ [C_4]: $z_2 \neq 0$ ∧
[C_5]: (∀m_1 m_2. dgtl_scnd_odr_fltr x y p q m_1 m_2)

$$\Rightarrow \frac{\text{z\_2d\_trnsfm y } z_1\ z_2}{\text{z\_2d\_trnsfm x } z_1\ z_2} = \frac{1}{1 - \frac{1}{4} * {z_2}^{-1} + \frac{1}{4} * {z_1}^{-2} - \frac{1}{2} * {z_1}^{-2} * {z_2}^{-1}}$$

Condition C_1 provides the ROC of the dynamical model of the 2^{nd}-order digital IPF. Condition C_2 ensures the first quadrant conditions on the input (x) and output (y) 2D arrays. Conditions C_3 and C_4 assert the non-zero condition for the variables z_1 and z_2. Condition C_5 presents the dynamical model of the 2^{nd}-order digital filter captured by Equation (6). The transfer function of the IPF is verified based on these assumptions as the conclusion of the theorem. The verification of Theorem 1 depends on the formal development of the 2D z-transform described in Section 3.

Next, we use the transfer function to formally verify the stability of the 2^{nd}-order 2D digital IPF as follows:

Theorem 2. $\vdash_{thm}$ ∀z_1 z_2. [C_1]: $z_1 \neq 0$ ∧ [C_2]: $z_2 \neq 0$ ∧

$$\Rightarrow \text{is\_stbl\_dgtl\_fltr} \left(\frac{1}{1 - \frac{1}{4} * {z_2}^{-1} + \frac{1}{4} * {z_1}^{-2} - \frac{1}{2} * {z_1}^{-2} * {z_2}^{-1}} \right)$$

Conditions C_1 and C_2 assert the non-zero condition for the variables z_1 and z_2. Finally, the conclusion models the stable 2^{nd}-order IPF. The verification of the above theorem is based on formalization of the stability, provided in Section 4.

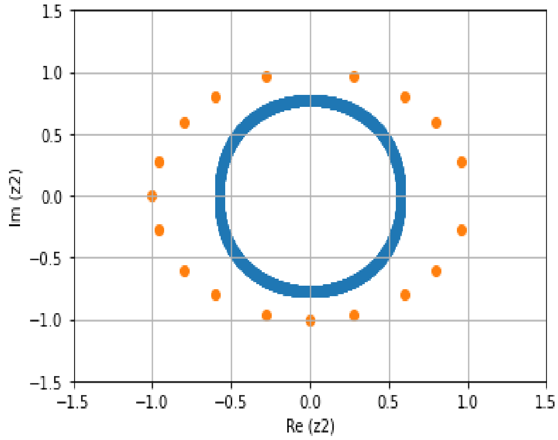

Fig. 2: Stability of the 2^{nd}-order Digital IPF on Root Map

Finally, we implement Condition (a) of the stability of the 2^{nd}-order digital IPF (Theorem 2) using the Python language. For this, we implement the characteristic equation $1 - \frac{1}{4}{z_2}^{-1} + \frac{1}{4}{z_1}^{-2} - \frac{1}{2}{z_1}^{-2}{z_2}^{-1} = 0$ on the complex plane z_2 for $z_1 = e^{i\omega_1}, \omega_1 \in [0, \pi]$. In the case of the 2^{nd}-order digital IPF (Figure 2), the presence of poles inside the unit circle contributes to the stability of the filter. Similarly, we can implement Condition (b), which alongside Condition (a) ensure the stability of the corresponding filter.

The main novelty of our results is that the generic nature of the verified properties, i.e., all theorems are verified for the universally quantified variables and functions. For example, we have formalized the dynamical model of the 2^{nd}-order filter using the (L_1, L_2)-order linear differential equations by specializing the generalized gains $(\alpha(l_1, l_2)$, $\beta(k_1, k_2)$ to some particular values. Another positive aspect of our formal stability analysis, presented in this paper, is the assurance of explicit presence of all the required assumptions along with the theorem that are often ignored in the traditional methods. These advantages are obtained at the cost of significant involvement of a user in the formal stability analysis, due to the usage of an interactive theorem proving tool. To reduce this user intervention, we proposed several simplifiers, such as DFRNC_EQU_TAC and TRNSFR_FNCTN_TAC[4] that significantly reduce the user guidance in the reasoning process.

6 Conclusions

Stability of a digital IPF is one of their important characteristics ensuring a stable output for a bounded input. We advocate using interactive theorem proving for performing stability analysis of these filters. In this regard, we formalized

[4] `https://save.seecs.nust.edu.pk/fsadipf/`

a dynamical model of the digital IPF and used the 2D z-transform to formally conduct the stability analysis. Finally, as a case study, we performed the stability analysis of a 2D digital IPF. In future, we aim to model the 2D convolution to develop formal reasoning support for systems-of-systems involving various image processing tasks [3].

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